MATH 504
EXERCISES 3

A. ZEYTİN

Unless otherwise stated G is a group.

(1) For any prime number p dividing the order of G, determine all Sylow p-subgroup of G where
- $G = \mathfrak{S}_3$
- $G = \mathfrak{S}_4$

(2) Show that for some p dividing the order of G, G has a normal Sylow p-subgroup (i.e G is not simple) when
- $|G| = 15$.
- $|G| = 20$.
- $|G| = 45$
- $|G| = 48$.
- $|G| = 56$.
- $|G| = 105$
- $|G| = 132$
- $|G| = 200$
- $|G| = 255$
- $|G| = 312$.
- $|G| = 351$.
- $|G| = p^r m$; where p is a prime number, r is a positive integer and $m < p$.
- $|G| = pqr$; where p, q and r are distinct prime numbers.

(3) Let G be a group and N be a normal p-subgroup of G. Show that N is contained in every element of $\mathrm{Syl}_p(G)$. In a somewhat converse manner let M be the intersection of all the Sylow p-groups of G. Show that M is a normal subgroup of G. Show that any other normal p-group in G is contained in M.

(4) Let $P \in \mathrm{Syl}_p(G)$ be a normal subgroup for some prime number p. For any $H \leq G$ show that $P \cap H$ is the unique Sylow p-subgroup of H.

(5) Prove that if $N \trianglelefteq G$ then for any prime number p dividing both $|G|$ and $|N|$ we have $n_p(G/N) \leq n_p(G)$.

(6) Let G be a finite group. Prove that G is nilpotent if and only if it has a *normal* subgroup of each order dividing $|G|$. Deduce that G is cyclic if and only if it has a *unique* subgroup or each order dividing $|G|$.

(7) Let $G = D_{2\cdot 5}$. Show that $C(G) = \{e\}$. Deduce that G is not nilpotent. Show more generally that if n is a positive odd integer, then $D_{2\cdot n}$ has trivial center and thus not solvable.

(8) Decide whether D_8 is solvable or not.

(9) Show that if $G/C(G)$ is nilpotent then G is nilpotent.

(10) Prove that a finite group G is nilpotent if and only if for $a, b \in G$ having relatively prime orders one has $ab = ba$.

(11) Find derived subgroups of $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

(12) Show indeed that
- if G is a solvable group and $\varphi\colon G \to H$ a group homomorphism, then $\varphi(G)$ is solvable. Deduce that if φ is an epimorphism, then H is solvable.
- if $N \trianglelefteq G$ and both N and G/N are solvable, then G is solvable.