MATH 504
EXERCISES 4

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Unless otherwise stated R is a ring.

(1) Let X be a non-empty set and $\mathcal{P}(X)$ is the set of all subsets of X. For any two elements $A, B \in \mathcal{P}(X)$ we define

$$A + B := (A - B) \cup (B - A) \quad \text{and} \quad A \cdot B := A \cap B$$

- ▶ Show that $\mathcal{P}(X)$ endowed with these two operations become a ring.
- ▶ Show that $\mathcal{P}(X)$ is commutative.
- ▶ Show that $\mathcal{P}(X)$ has identity.
- ▶ Show that for any subset A of $P(X)$ we have $A^2 = A$. Such rings, that rings for which $x^2 = x$ for all $x \in R$ are called *Boolean rings*.
- ▶ Investigate what happens if we define $A + B := A \cup B$.

(2) Let R and S be two rings.
- ▶ Define proper operations (addition and multiplication) on $R \times S$ so that $R \times S$ becomes a ring.
- ▶ Show that the following sets are subrings of $R \times S$
  - i. $\{(r, 0) : r \in R\}$
  - ii. $\{(0, s) : s \in S\}$
- ▶ Prove that $R \times S$ is commutative if and only if both R and S are commutative.
- ▶ Show that $R \times S$ is has an identity if and only if both R and S has identity.
- ▶ Show that even if R and S are integral domains $R \times S$ is not an integral domain.

(3) Let R be a ring with unity, i.e. $1 \in R$. Show that the commutativity of addition is *forced* by distributivity. Hint: For any $a, b \in R$ compute $(1 + 1)(a + b)$ in two different ways.

(4) For any square-free integer D we let

$$\mathbf{Q}(\sqrt{\delta}) := \left\{ a + b\sqrt{\delta} \mid a, b \in QQ \right\}.$$

- ▶ Show that $\mathbf{Q}(\sqrt{\delta})$ is a field. Decide what happens if δ is not square-free.
- ▶ Show that $\mathbf{Z}[\sqrt{\delta}] := \left\{ a + b\sqrt{\delta} \mid a, b \in \mathbf{Z} \right\}$ is a subring of $\mathbf{Q}(\sqrt{\delta})$. When $\delta = -1$ $\mathbf{Z}[\sqrt{-1}]$ is called the ring of Gaussian integers.

(5) Decide which of the following sets are subrings of the ring of all function from $[-1, 1]$ to $\mathbf{R}$, denoted by $F([-1, 1], \mathbf{R})$:
- ▶ $\{f \in F([-1, 1], \mathbf{R}) \mid f(q) = 0 \text{ for all } q \in \mathbf{Q}\}$
- ▶ $\{f \in F([-1, 1], \mathbf{R}) \mid f \text{ f is a polynomial}\}$
- ▶ $\{f \in F([-1, 1], \mathbf{R}) \mid f \text{ has infinitely many zeroes}\}$
- ▶ $\{f \in F([-1, 1], \mathbf{R}) \mid \lim_{x \to 0} f(x) \text{ exists}\}$
- ▶ $\{f \in F([-1, 1], \mathbf{R}) \mid \lim_{x \to 0} f(x) = 0\}$
- ▶ $\{f \in F([-1, 1], \mathbf{R}) \mid \lim_{x \to 0} f(x) = 1\}$

(6) Let $\alpha \in R$ be a nilpotent element of R, that is there is some natural number m so that $\alpha^m = 0$.
- ▶ Show that α is either 0 or a zero divisor.
- ▶ For any $r \in R$ show that $r\alpha$ is also a nilpotent element.
- ▶ If β is another nilpotent element of R, then $\alpha - \beta$ is also nilpotent.
- ▶ Show that if $1 \in R$, then $1 + \alpha$ is a unit. Prove, more generally that nilpotent+unit = unit.

(7) Show that a ring homomorphism $\varphi \colon R \to S$ is injective if and only in $\ker(\varphi) = \{0\}$.

(8) Let $R = C^\infty(\mathbf{R}, \mathbf{R})$ be the ring of all infinitely many times differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ Define :

$$\begin{aligned} \varphi \colon R &\longrightarrow R \\ f(t) &\longmapsto \frac{d}{dt} f(t) \end{aligned}$$

Decide whether φ is a ring homomorphism.

(9) Give an example of a ring homomorphism $\varphi\colon R \to R'$ where R is a ring with 1 but $\varphi(1)$ is not a unit of R'.

(10) Find a subring of $\mathbf{Z} \times \mathbf{Z}$ that is not an ideal.

(11) Let $\varphi\colon R \to R'$ be a ring homomorphism, I an ideal of R and I' and ideal of R'.
- Show that $\varphi(I)$ is an ideal of $\mathrm{im}(()\varphi)$.
- Show that $\varphi^{-1}(I')$ is an ideal of R.
- If I is a maximal/prime ideal of R decide whether $\varphi(I)$ is a maximal/prime ideal of $\mathrm{im}(()\varphi)$.
- If I' is a maximal/prime ideal of R' decide whether $\varphi^{-1}(I)$ is a maximal/prime ideal of R.
- Give an example of a ring homomorphism $\varphi\colon R \to R'$ so that $\varphi(I)$ is not an ideal of R'.

(12) Let R be a commutative ring with 1 and let $a \in R$ be a fixed but arbitrary element. Show that the set $\mathrm{Ann}(a) = \{r \in R\colon ra = 0\}$ is an ideal of R.

(13) Show that the collection of all nilpotent elements of a ring R form an ideal, called the *nilradical* of R.

(14) Find all ideals of the following rings. Decide which of them are maximal/prime?
- $\mathbf{Z}/6\mathbf{Z}$,
- $\mathbf{Z}/12\mathbf{Z}$,
- $\mathbf{Z}/18\mathbf{Z}$,
- $\mathbf{Z}/6\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.

(15) Let I, J be two ideals of a ring R. Show that the set

$$I : J := \{r \in R\colon rj \in I \text{ for all } j \in J\}$$

is an ideal of R. Compute $I : J$ where
- $R = \mathbf{Z}$, $I = 6\mathbf{Z}$, $J = 18\mathbf{Z}$
- $R = \mathbf{Z}$, $I = 18\mathbf{Z}$, $J = 6\mathbf{Z}$
- $R = \mathbf{Z}$, $I = 8\mathbf{Z}$, $J = 12\mathbf{Z}$
- $R = \mathbf{Z}$, $I = 12\mathbf{Z}$, $J = 8\mathbf{Z}$

(16) Let $R = M^{2\times 2}(\mathbf{Z}/2\mathbf{Z})$.
- Determine the number of elements in R.
- Determine all ideals of R.

(17) Complete the proofs of 3rd and 4th isomorphism theorems.