MATH 504
EXERCISES 6

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Here R unless otherwise stated R is a ring.

(1) Let R, S be two rings and $\varphi\colon R \to S$ is a ring homomorphism. Show that every S module M can also be endowed with an R-module structure. Interpret this result in terms of vector spaces over fields.

(2) Let M be an R-module and M' be an R-submodule of M. Show that M/M' can be endowed with an R-module structure.

(3) Let R be a ring, M and N be two R modules. Let $\varphi\colon M \to N$ be an R-module homomorphism.
- ▶ Show that $\ker(\varphi)$ is an R-submodule of M and $\mathrm{im}(()\varphi)$ is an R-submodule of N.
- ▶ Show that if N' is any R-submodule of N then $\varphi^{-1}(N')$ is a submodule of M.
- ▶ Let M' be an R-submodule of M. Decide whether $\varphi(M')$ is necessarily an R-submodule of N.
- ▶ Show that $\ker(\varphi) = \{0\}$ if and only if φ is injective.
- ▶ Show that φ is an isomorphism if and only if there is an R-module homomorphism $\psi\colon N \to M$ such that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$.
- ▶ Show that $M/\ker(\varphi) \cong \mathrm{im}(()\varphi)$.

(4) Let M be an R-module and M' and M'' two R-submodules of M.
- ▶ Show that $M' \cap M''$ is an R-submodule of M.
- ▶ Show that the set $M' + M'' := \{m' + m''\colon m' \in M' \text{ and } m'' \in M''\}$ is an R-submodule of M.
- ▶ Show that $M'/(M' \cap M'') \cong (M' + M'')/M''$.
- ▶ If $M'' \subset M'$, then show that M'/M'' is an R-submodule of M/M''.
- ▶ If $M'' \subset M'$ show that $(M/M'')/(M'/M'') \cong M/M'$. (cancellation law!)

(5) Let R be commutative ring with 1 and consider the ring $R[x]$ with its canonical R-module structure. Show that for any fixed but arbitrary $f(x) \in R[x]$ the map $p(x) \mapsto f(x)\,p(x)$ defines an R-module homomorphism from $R[x]$ to itself. Show on the other hand that this is NOT a ring homomorphism.

(6) Let $M = \mathbf{Z}$, $N = \mathbf{Z}/24\mathbf{Z}$. Determine the modules $\mathrm{Hom}_{\mathbf{Z}}(M, N)$ and $\mathrm{Hom}_{\mathbf{Z}}(N, M)$. How many elements do they have?

(7) Let M and N be two R-modules. Consider the following two sequences :

$$0 \to M \xrightarrow{\iota_1} M \oplus N \xrightarrow{\pi_2} N \to 0$$
$$0 \to NN \xrightarrow{\iota_2} M \oplus N \xrightarrow{\pi_1} M \to 0$$

where the maps are defined as :

$$\begin{aligned} \iota_1\colon M &\to M \oplus N \\ m &\mapsto (m, 0) \end{aligned} \qquad\qquad \begin{aligned} \iota_2\colon N &\to M \oplus N \\ n &\mapsto (0, n) \end{aligned}$$

and

$$\begin{aligned} \pi_1\colon M \oplus N &\to M \\ (m, n) &\mapsto m \end{aligned} \qquad\qquad \begin{aligned} \pi_2\colon M \oplus N &\to N \\ (m, n) &\mapsto n \end{aligned}$$

- ▶ Show that all of the above maps are R-module homomorphisms.
- ▶ Show that the above two sequence are short exact.

(8) Let M be an R-module and M' be an R-submodule of M. Show that the sequence

$$0 \to M' \to M \to M/M' \to 0$$

is a short exact sequence.

(9) Let M, M', M'' and N, N', N'' be R-modules and let

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$$
$$0 \to N' \xrightarrow{\varphi'} N \xrightarrow{\psi'} N'' \to 0$$

be two short exact sequences. Assume that there are R-module homomorphisms $\alpha\colon M' \to N'$, $\beta\colon M \to N$, $\gamma\colon M'' \to N''$ so that

$$\beta \circ \varphi = \varphi' \circ \alpha \quad \text{and} \quad \gamma \circ \psi = \psi' \circ \beta.$$

In summary, we have :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & N' & \xrightarrow{\varphi'} & N & \xrightarrow{\psi'} & N'' & \longrightarrow & 0 \end{array}$$

- ▶ Show that if α and γ are injective then β is injective.
- ▶ Show that if α and γ are surjective then β is surjective.
- ▶ Deduce that if α and γ are isomorphisms then β is and isomorphism.

(10) Let M be an R-module and $m \in M$ be a fixed but arbitrary element of M.

- ▶ Show that the set $I(m) = \{r \in R\colon r \cdot m = 0\}$ is an ideal of R.
- ▶ For $M = \mathbf{Z}/24\mathbf{Z}$ considered as a Z-module, compute $I(\overline{2})$, $I(\overline{3})$, $I(\overline{6})$, $I(\overline{12})$, $I(\overline{14})$.
- ▶ For $M = \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$ considered as a Z-module, compute $I((\overline{2},\overline{1}))$, $I((\overline{2},\overline{2}))$, $I((\overline{2},\overline{3}))$, $I((\overline{2},\overline{6}))$, $I((\overline{2},\overline{9}))$.
- ▶ For $M = \mathbf{Z}/ \times \mathbf{Z}/6\mathbf{Z}$ considered as a Z-module, compute $I((\overline{2},\overline{1}))$, $I((\overline{2},\overline{2}))$, $I((\overline{2},\overline{3}))$, $I((\overline{2},\overline{6}))$, $I((\overline{2},\overline{9}))$.

An element $m \in M$ is called a torsion element of M if $I(m) \neq \{0\}$. Let $T(M)$ denote the set of all torsion elements of M.

- ▶ Determine all torsion elements of $\mathbf{Z}/24\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$ and $\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$, that is determine $T(\mathbf{Z}/24\mathbf{Z})$, $T(\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z})$ and $T(\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z})$.
- ▶ Show that $T(M)$ is an R-submodule of M whenever R is an integral domain. Show, by an explicit example that $T(M)$ is not necessarily a submodule if R is not an integral domain.
- ▶ Let $\varphi\colon M \to N$ be an R-module homomorphism between the R-modules M and N. Show that $\varphi(T(M)) \subset T(N)$.
- ▶ Show that if

  $$0 \to M' \to M \to M'' \to 0$$

  is an exact sequence of R-modules M, M' and M'', then the following torsion version of the above sequence is exact:

  $$0 \to T(M') \to T(M) \to T(M'') \to 0$$