Université Galatasaray, Département de Mathématiques
2017 - Fall Semester – Math 504 - Advanced Algebra
Final, 10 January 2018 – Ayberk Zeytin, 180 min.s
Name & Surname: ______________________________ **Sign:** ______________

Question:	1	2	3	4	Total
Points:	16	6	4	6	32
Score:					

Question 1 (16 points)
Let G be a group and let $\text{Aut}(G)$ denote the group of automorphisms of G. Recall that for any $g \in G$ the map

$$\begin{aligned} \varphi_g \colon G &\to G \\ x &\mapsto g\,x\,g^{-1} \end{aligned}$$

is called an inner automorphism, and the set of all inner automorphisms of G, denoted by $\text{Inn}(G)$, is a subgroup of $\text{Aut}(G)$. Define :

$$G_1 = G \text{ and } G_{n+1} = \text{Aut}(G_n) \text{ for } n \in \mathbf{N}$$

and consider the family of maps

$$\begin{aligned} \pi_n \colon G_n &\to G_{n+1} \\ g &\mapsto \pi_n(g); \end{aligned}$$

where $\pi_n(g) \colon G_n \to G_n$ is defined by sending $x \in G_n$ to $g\,x\,g^{-1}$.

(a) (2 points) Show that $\pi_n \colon G_n \to G_{n+1}$ is a group homomorphism and hence we have a sequence of maps :

$$G_1 = G \xrightarrow{\pi_1} G_2 = \text{Aut}(G) \xrightarrow{\pi_2} G_3 \xrightarrow{\pi_3} \cdots \tag{1}$$

(b) (2 points) Show that $\pi_n(G_n)$ is a normal subgroup of G_{n+1}.

(c) (2 points) Show that whenever $Z(G) = \{e\}$, the maps π_n are injective.

(d) (2 points) Show that $C_{G_{n+1}}(\pi_n(G_n)) = \{e\}$.

For the remainder of the problem, we set $G = \mathfrak{S}_4$, the symmetric group on $\{1, 2, 3, 4\}$.

(e) (2 points) Show that any automorphism φ of G permutes the Sylow 3-subgroups of G, i.e. if $\varphi\colon G \to G$ an element of $\mathrm{Aut}(G)$, and P is any Sylow 3 subgroup of G, then $\varphi(P)$ is a Sylow 3-subgroup of G.

(f) (2 points) Show that if $\varphi \in \mathrm{Aut}(\mathsf{G})$ fixes each Sylow 3-subgroup, then φ is the identity.

(g) (2 points) Deduce that every automorphism of G is an inner automorphism.

(h) (2 points) For the group $G = \mathfrak{S}_4$ explicitly determine the sequence given in

$$G_1 \to G_2 \to G_3 \to \cdots$$

Question 2 (6 points)

Let p be a prime number and $G = GL_2(\mathbf{Z}/p\mathbf{Z})$ be the group of invertible 2×2 matrices with entries from the field $\mathbf{Z}/p\mathbf{Z}$.

(a) (2 points) Show that the order of G is $(p^2 - 1)(p^2 - p)$. (Hint: Try to count the number of linearly independent pairs of vectors.)

(b) (2 points) Determine the number of Sylow p-subgroups of G.

(c) (2 points) Explicitly describe a Sylow p-subgroup of G.

Question 3 (4 points)

Let S be a multiplicative subset of a non-trivial commutative ring R.

(a) (2 points) Show that if R is an integral domain, then so is $\mathsf{S}^{-1}\mathsf{R}$.

(b) (2 points) For $\mathsf{S} = \{2, 4\} \subset \mathsf{R} = \mathbf{Z}/6\mathbf{Z}$ what is $\mathsf{S}^{-1}\mathsf{R}$. Relate this to part (a).

Question 4 (6 points)

Let R be a commutative ring with 1 and let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence of R-modules. Assume that there are elements r, s in R so that $(r, s) = R$. Suppose that $rA = \{0\}$ and $sC = \{0\}$.

(a) (2 points) Show multiplication by r map defines an R-module isomorphism from C to C, that is, the map :

$$\begin{aligned} m_r \colon C &\to C \\ c &\mapsto rc \end{aligned}$$

is an isomorphism of R-modules.

(b) (2 points) Show that $rB = \{r\,b \colon b \in B\}$ is a submodule of B and the restriction of the ψ to rB in an R-module isomorphism from rB to C.

(c) (2 points) Show that $\mathsf{B} \cong \mathsf{A} \oplus \mathsf{C}$.