Université Galatasaray, Département de Mathématiques
2017 - Fall Semester – Math 504 - Advanced Algebra
Mid Term Exam, 15 November 2017 – Ayberk Zeytin, 120 minutes
Name & Surname: ____________________________________ **Sign:** ______________

Question:	1	2	3	4	Total
Points:	2	2	4	8	16
Score:					

Question 1 (2 points)
Decide which of the following groups are isomorphic :

$$\mathbf{Z}/24\mathbf{Z},\ \mathbf{Z}/4\mathbf{Z}\times\mathbf{Z}/6\mathbf{Z},\ \mathfrak{S}_4,\ \mathrm{A}_4\times\mathbf{Z}/2\mathbf{Z},\ \mathbf{Z}/8\mathbf{Z}\times\mathbf{Z}/3\mathbf{Z},\ \mathrm{D}_{24},\ \mathrm{D}_{12}\times\mathbf{Z}/2\mathbf{Z}$$

Question 2 (2 points)

Let G be a finite group. Prove that any subgroup of index equal to the smallest prime dividing $|\mathsf{G}|$ is normal. (`Hint:` Consider an action of G on the coset space with respect to the subgroup, and find its kernel.)

Question 3 (4 points)

Let $\varphi\colon \mathsf{G} \to \mathsf{H}$ be a homomorphism of groups. Let $\mathsf{S}(\mathsf{G})$ denote the set of all subgroups of G and $\mathsf{S}(\mathsf{H})$ denote the set of all subgroups of H. Define

$$\begin{aligned} \varphi^*\colon \mathsf{S}(\mathsf{G}) &\to \mathsf{S}(\mathsf{H}) \\ \mathsf{K} &\mapsto \varphi(\mathsf{K}) \end{aligned}$$

(a) (2 points) Show that if φ is an isomorphism then φ^* defines a bijection between $\mathsf{S}(\mathsf{G})$ and $\mathsf{S}(\mathsf{H})$.

(b) (2 points) Assume that φ is just an epimorphism. Find a proper subset, say $\mathsf{S}_\varphi(\mathsf{G})$, $\mathsf{S}(\mathsf{G})$ so that φ^* is a bijection between $\mathsf{S}_\varphi(\mathsf{G})$ and $\mathsf{S}(\mathsf{H})$.

Question 4 (8 points)
Let G be a group of order $2^3\,7^2\,11$.

(a) (2 points) Let p and q be distinct prime numbers and H be a group of order $\mathsf{p}^2\mathsf{q}$ having a unique Sylow q-subgroup. Show that H is abelian if $1 \not\equiv \mathsf{q} \pmod{\mathsf{p}}$.

(b) (2 points) Show that G has a subgroup of order 77. Denote this subgroup by H_{77}. (`Hint:` Use preceeding exercise and the fact that an abelian group has a subgroup of size n for each n diving its order.)

(c) (2 points) Show that G has a subgroup of order 7 whose normalizer has index dividing 8. Denote this subgroup by H_7. (`Hint:` The correct candidate is the Sylow 7-subgroup of H_{77}. Try to determine the order of its normalizer.)

(d) (2 points) Show that G is not simple. (`Hint:` Consider the permutation representation induced by the action of G on $\mathsf{N}_{\mathsf{G}}(\mathsf{H}_7)$. G has elements of order 11!)