Université Galatasaray, Département de Mathématiques
2017 - Fall Semester – Math 504 - Advanced Algebra
Mid Term Exam, 18 December 2017 – Ayberk Zeytin, 150 min.s
Name & Surname: ______________________ **Sign:** __________

Question:	1	2	3	4	Total
Points:	4	6	8	10	28
Score:					

Question 1 (4 points)
Let G be a nilpotent group and $\varphi\colon \mathsf{G} \to \mathsf{H}$ be a surjective group homomorphism.

(a) (2 points) Show that H is also nilpotent.

(b) (2 points) Deduce that if φ is non-trivial, i.e. $\mathsf{H} \neq \{e\}$, then H has non-trivial center.

Question 2 (6 points)

Let R, R' be two commutative rings with identity.

(a) (2 points) Let $\varphi \colon \mathsf{R} \to \mathsf{R}'$ be a surjective ring homomorphism and R be a PID. Show that R' is an integral domain if and only if R' is a field.

(b) (2 points) Deduce that if k is a field, then any ring homomorphism $\psi \colon k[x] \to \mathbf{Z}$ is must be trivial.

(c) (2 points) Construct infinitely many distinct homomorphisms from $\mathbf{Q}[x]$ to $\mathbf{Q}$.

Question 3 (8 points)

Let R be the ring of all continuous functions from $[0, 1]$ to $\mathbf{R}$.

(a) (2 points) Determine the units of R.

(b) (2 points) For any fixed $a \in [0, 1]$, let I_a denote the set of all elements of R which vanish at a; that is

$$I_a := \{f\colon [0,1] \to \mathbf{R} \mid f(a) = 0\}.$$

Show that I_a is a maximal ideal of R.

(c) (2 points) Show that the set :

$$\{f \in R \colon f(1/504) = 0 \text{ and } f(1/501) = 0\}$$

is an ideal of R. Is it prime?

(d) (2 points) Show that any maximal ideal of R is of the form I_a for some $a \in [0, 1]$.

Question 4 (10 points)

Let $d > 3$ be a square-free integer and set $R = \mathbf{Z}[\sqrt{-d}]$.

(a) (2 points) Show that the function $N \colon R \to \mathbf{Z}$ defined by $N(a + b\sqrt{-d}) := a^2 + db^2$ is a norm on R.

(b) (2 points) Show that N is multiplicative, i.e. $N(\alpha\,\beta) = N(\alpha)\,N(\beta)$ for any $\alpha, \beta \in R$.

(c) (2 points) Show that 2, $\sqrt{-d}$ and $1 + \sqrt{-d}$ are all irreducible in R.

(d) (2 points) Prove that R is not a UFD.

(e) (2 points) Give an explicit ideal in R which is not principal. (`Hint:` Your candidate can be a maximal ideal. Use (c).)