MATH 532
EXERCISES 2

A. ZEYTİN

Unless otherwise stated K and k are fields and K/k is a field extesion.

(1) Determine the splitting field of $X^3 - 2$. Does this field contain $\sqrt{-1}$?

(2) Let p be a prime number. For the polynomial $f(X) = \frac{X^p - 1}{X-1} \in \mathbf{Q}[X]$ show that if ζ_p is a root then the remaining roots are $\zeta_p^2, \dots \zeta_p^{p-1}$. Deduce the splitting field of f.

(3) Let k be a field of characteristic p; where p is a prime number. Let $f(X) = X^p - X - a$ for some $a \in k$.
- ▶ If α is a root of f then show that $\alpha + 1$ is also a root of f.
- ▶ Determine the remaining roots of f.
- ▶ What is a splitting field for f.

(4) Let k be a field of characteristic diffrent from 2 and let K be an extension of k of degree 2, that is $[K : k] = 2$. Consider the set:
$$Q(K) = \{\alpha \in K^\times \mid \alpha = a^2 \text{ for some a in } K\}.$$
- ▶ Show that $Q(K)$ is a multiplicative subgroup of $K^\times$.
- ▶ Show that if K and K' are two quadratic extensions of k, then K and K' are isomorphic if and only if $Q(K)$ and $Q(K')$ are isomorphic (as multiplicative groups).
- ▶ Show that there are infinitely non-isomorphic quadratic extensions of $\mathbf{Q}$.
- ▶ Show that up to isomorphism, there is a unique extension of $\mathbf{Z}/p\mathbf{Z}$ of degree 2.

(5) Prove the usual differentiation rules for the formal derivative. Namely, for $f(X), g(X) \in k[X]$, prove that :
- $(f + g)' = f' + g'$
- $(f\,g)' = f'\,g + f\,g'$