MATH 532
EXERCISES 3

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Unless otherwise stated K and k are fields and K/k is a field extesion.

(1) Show that if p and q are two distinct prime numbers then the fields $\mathbf{Q}(\sqrt{p})$ and $\mathbf{Q}(\sqrt{q})$ are not isomorphic.

(2) Let k be a field and consider the extension $k(t)/k$.
- ▶ Show that $\sigma : t \mapsto t+1$ defines an automorphism of $k(t)$.
- ▶ Determine the fixed field of σ.

(3) Let K/k be an extension and $\varphi\colon K \to K'$ be a field isomorphism. Let $k' = \varphi(k)$.
- ▶ Show that k' is a subfield of K'.
- ▶ Show that the map

$$\pi\colon \operatorname{Aut}(K/k) \to \operatorname{Aut}(K'/k')$$
$$\sigma \mapsto \varphi\,\sigma\,\varphi^{-1}$$

defines a group isomorphism.

(4) Let k be a field of characteristic 0. Consider the extensions $K = k(X^2)$ and $K' = k(X^2 - X)$ as subextensions of $k(X)$. Show that $K \cap K' = k$. Hint: Consider $\operatorname{Aut}(k(X)/K)$ and $\operatorname{Aut}(k(X)/K')$. They both contain two elements. What is the order of the composition?

(5) Let p be an odd prime number and let $\zeta_p = \exp(2\pi\sqrt{-1}/p)$. Set $K = \mathbf{Q}(\zeta_p)$.
- ▶ Find the minimal polynomial of ζ_p and show that the remaining roots are $\exp(k2\pi\sqrt{-1}/p)$ for $2 \leq k \leq p-1$.
- ▶ Determine the automorphism group of the extension $K/\mathbf{Q}$ and deduce that the extension $K/\mathbf{Q}$ is Galois.
- ▶ Show that $\operatorname{Gal}(K/\mathbf{Q})$ contains a unique subgroup of index 2, denoted H henceforth.
- ▶ Define $\alpha = \sum_{\sigma \in H} \sigma(\zeta_p)$ and $\beta = \sum_{\sigma \in G \setminus H} \sigma(\zeta_p)$. Show that $\alpha, \beta \in \operatorname{Fix}(H)$.
- ▶ For any $\sigma \in (G \setminus H)$ show that $\sigma(\alpha) = \beta$ and $\sigma(\beta) = \alpha$.
- ▶ Deduce the minimal polynomial of α (and hence β).
- ▶ Determine $\alpha\beta$ explicitly and show that $\operatorname{Fix}(H) = \mathbf{Q}(\sqrt{p})$.
- ▶ Make the above computations explicit when $p = 5$.

(6) Let $K = \mathbf{Q}(\sqrt{2}, \sqrt{3})$ and $L = K\left(\sqrt{(\sqrt{2}+2)(\sqrt{3}+3)}\right)$ considered as subextensions of the extension $\mathbf{R}/\mathbf{Q}$.
- ▶ Show that the extension $K/\mathbf{Q}$ is Galois with $\operatorname{Gal}(K/\mathbf{Q}) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.
- ▶ Show that $L/\mathbf{Q}$ is Galois with $\operatorname{Gal}(L/\mathbf{Q})$ being the quaternion group.

(7) Determine the minimal polynomials of the following elements:
- ▶ $\sqrt{2} + \sqrt{3}$
- ▶ $1 + \sqrt[3]{2} + \sqrt[3]{4}$
- ▶ $\zeta_p + \zeta_p^{p-1}$ (or more generally $\zeta_p^i + \zeta_p^{p-i}$)

(8)
- ▶ Show that the polynomial $f(X) = X^4 - 2X^2 - 2$ is irreducible in $\mathbf{Q}[X]$
- ▶ Show that $\alpha_1 = \sqrt{1+\sqrt{3}}$, $\alpha_2 = -\alpha_1$, $\alpha_3 = \sqrt{1-\sqrt{3}}$, $\alpha_4 = -\alpha_3$ are the roots of $f(X)$.
- ▶ Show that the fields $\mathbf{Q}(\alpha_1)$ and $\mathbf{Q}(\alpha_3)$ are distinct subextensions of a splitting field of K.
- ▶ Show that $\mathbf{Q}(\alpha_1) \cap \mathbf{Q}(\alpha_3) = \mathbf{Q}(\sqrt{3})$
- ▶ Show that $\mathbf{Q}(\alpha_1)/\mathbf{Q}(\sqrt{3})$ is a Galois extension. What is its Galois group?
- ▶ Show that $\mathbf{Q}(\alpha_3)/\mathbf{Q}(\sqrt{3})$ is a Galois extension. What is its Galois group?
- ▶ Find the Galois group of a splitting field for $f(X)$.

(9)