MATH 532
EXERCISES 5

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Unless otherwise stated K and k are fields and K/k is a field extension.

(1) Let $\mathcal{D}$ be a semi-group with unique factorization and let $a, b, c \in \mathcal{D}$. Prove that :
- ▶ if ab is divisible by c and b is relatively prime to c, then a is divisible by c.
- ▶ if c is divisible by both a and b which are relatively prime, then c is divisible by their product ab.
- ▶ if a product ab is divisible by a prime element $p \in \mathcal{D}$, then at least one of a or b is divisible by p.

(2) Let $\mathcal{O}$ be a ring with a divisor theory $\mathcal{D}$. Explicitly determine which properties of a valuation fails when one considers a valuation ν_a associated with an element $a \in \mathcal{D}$ which is not prime.

(3) Let $\nu_p : \mathcal{O} \to \mathbf{Z}_{\geq 0}$ be the valuation associated to a prime divisor $p \in \mathcal{D}$ on an integral domain $\mathcal{O}$ endowed with a divisor theory $\mathcal{D}$.
- ▶ Show that $\nu_p(\alpha\beta) = \nu_p(\alpha) + \nu_p(\beta)$
- ▶ Show that $\nu_p(\alpha + \beta) \geq \min\{\nu_p(\alpha), \nu_p(\beta)\}$
- ▶ Let K be the field of fractions of $\mathcal{O}$. Extend the valuation ν_p to K via defining $\nu_p(\xi) = \nu_p(\alpha/\beta) := \nu_p(\alpha) - \nu_p(\beta)$, for $\alpha, \beta \in \mathcal{O}$. Verify that the extended ν_p is well-defined (i.e. independent of the choice of generators chosen for the class α/β) and satisfies all the axioms of a valuation.
- ▶ Show that distinct prime divisors $p \in \mathcal{D}$ give rise to distinct prime divisors.

(4) Let k be an algebraically closed field.
- ▶ Show that all irreducible polynomials in $k[X]$ are linear. Deduce that there is a one to one correspondence between k and irreducible polynomials in $k[X]$.
- ▶ Show that for each irreducible polynomials $X - a \in k[X]$ the function defined as the function :

$$\begin{aligned} \nu_a : k[X] &\to \mathbf{Z}_{\geq 0} \\ p(x) &\mapsto n; \end{aligned}$$

  where for any $p(X) \in k[X]$, n is the unique integer determined by $(X - a)^n | p(X)$ but $(X - a)^{n+1} \nmid p(X)$ defines a valuation on $k[X]$ and hence on $k(X)$.
- ▶ Whenever $k = \mathbf{C}$ show that this definition agrees with the definition of the order of vanishing of a function at a point $a \in \mathbf{C}$.