MATH 532
EXERCISES 5

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Unless otherwise stated K and k are fields and K/k is a field extension.

(1) Let ν be a valuation on a field k.
- ▶ Show that the set $\mathcal{O} = \{\alpha \in k \mid \nu(\alpha) \geq 0\}$ is a ring. This ring is called a valuation ring of k.
- ▶ Show that the set $\mathfrak{m} = \{\alpha \in k \mid \nu(\alpha) > 0\}$ is an ideal of k.
- ▶ Show that an element in $\mathcal{O}$ is a unit if and only if $\nu(\alpha) = 0$, hence the units of $\mathcal{O}$ are characterized by the property that $\nu(\alpha) = 1$.
- ▶ Show that there is an element $\pi \in \mathcal{O}$ with $\nu(\pi) = 1$. π is called a *uniformizer*.
- ▶ Show that any element $x \in \mathcal{O}$ can be written as $x = u\pi^n$ for some unit u of $\mathcal{O}$ and non-negative integer n. Deduce that $\mathcal{O}$ is a UFD.
- ▶ Deduce further that $\mathcal{O}$ is a PID. more precisely, show that any ideal I of $\mathcal{O}$ is of the form (π^n) for some non-negative integer n.
- ▶ Deduce that $\mathcal{O}/\mathfrak{m}$ is a field, called the *residue field*.
- ▶ Let p be an odd prime number and ν_p the associated valuation on $\mathbf{Q}$. Determine $\mathcal{O}$, $\mathfrak{m}$ and the residue field $\mathcal{O}/\mathfrak{m}$.
- ▶ Repeat the previous problem for a fixed monic irreducible polynomial $p(X) \in k[X]$; where k is an algebraically closed field.

(2) Let $K = k(X)$ be the field of rational functions with coefficients from the field k. Let $f(X)$ be an irreducible polynomial in $k[X]$.
- ▶ Show that every element $u(X) \in K$ can be written as $f^k(\alpha(X)/\beta(X))$ for some polynomials $\alpha(X), \beta(X) \in k[X]$ which are relatively prime to $f(X)$.
- ▶ Use the previous exercise to define the valuation of $u(X)$, denoted by $\nu_f(u)$ as k. Show that this definition gives an honest valuation on K.
- ▶ Exemplify the situation described above for various different fields.

(3) For any field k on the ring $k[X]$ define the valuation $\nu_\infty(\alpha(X)/\beta(X)) := \deg(\alpha(X)) - \deg(\beta(X))$. Show that ν_∞ defines a valuation on $k(X)$.