MATH 532
EXERCISES 7

A. ZEYTİN

(1) Find the p-adic norm and the p-adic expansion of :
   - ▶ $15, -1, -3$ in $\mathbf{Q}_5$
   - ▶ $6, 1/3$ in $\mathbf{Q}_3$
   - ▶ $1/3$ in $\mathbf{Q}_5$
   - ▶ $1/6, 1/11$ in $\mathbf{Q}_7$

(2) Find the p-adic expansion of $\frac{1}{2}$ in $\mathbf{Q}_p$ of an odd prime number p.

(3) Show that in $\mathbf{Q}_7[X]$ we have : $X^2 - 2 = (X-3)(X-4)$.

(4) Let $f(X) = X^3 - 2 \in \mathbf{Q}_5[X]$. Show that there is a unique cube root, α, of 2 in $\mathbf{Z}_5$ with $\alpha \equiv 3 \pmod 5$. Find first few terms of the canonical expansion of $\alpha \in \mathbf{Q}_5$.

(5) Recall that $\mathbf{Z}_p := \{\alpha \in \mathbf{Q}_p \,|\, |\alpha|_p \leq 1\}$.
   - ▶ Show that $\mathbf{Z}_p$ is a ring.
   - ▶ Show that the units in $\mathbf{Z}_p$ are precisely the sequences $(a_n)_{n=0}^{\infty}$ for which $a_0 \neq 0$ by constructing the inverse of such an element.
   - ▶ Show that $\mathbf{Z}_p$ is uncountable. (This means in particular that $\mathbf{Z}_p$ is much much bigger than $\mathbf{Z}$.)
   - ▶ Let $f(X) \in \mathbf{Z}[X]$. Show that if there is some element $\alpha \in \mathbf{Z}_p$ with $f(\alpha) = 0$, then for any positive integer r, there is some $x_r \in \mathbf{Z}/p^r\mathbf{Z}$ with the property that $f(x_r) = 0$ in $\mathbf{Z}/p^r\mathbf{Z}$.