Université Galatasaray, Département de Mathématiques
2018 - Spring Semester – Math 532 - Selected Topics in Algebraic Geometry
Final Exam, 28 May 2018 – Ayberk Zeytin, 180 minutes
Name & Surname: ______________________________ **Sign:** ____________

Question:	1	2	3	Total
Points:	14	2	10	26
Score:				

Throughout p denotes a prime number, $\mathbf{Q}_p$ denotes the field of p-adic numbers, and $\mathbf{Z}_p$ denotes the ring of p-adic integers.

Question 1 (14 points)

(a) (2 points) Write 5 as a p-adic number for $p = 2, 3$.

(b) (2 points) Write 5 as a p-adic number for $p > 5$.

(c) (2 points) Calculate $1/5$ as a 3-adic number in standard form. Is it a 3-adic integer (i.e. an element of $\mathbf{Z}_3$)?

(d) (2 points) Let

$$\alpha = 2 + 3 \cdot p + 5 \cdot p^2 + 2 \cdot p^3 + 3 \cdot p^4 + 5 \cdot p^5 + \dots$$

Which number does α represent in $\mathbf{Q}_p$ for $p > 5$?

(e) (2 points) Which number does α (of part (d)) represent in $\mathbf{Q}_p$ for $p = 2, 3$.

(f) (2 points) For $\sqrt{7} = a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + a_3 \cdot 3^3 + a_4 \cdot 3^4 + \ldots$, find $a_0, \ldots a_4$.

(g) (2 points) Show that there is no $\beta \in \mathbf{Q}_p$ so that $\beta^2 = p$. Deduce that $\mathbf{Q}_p(\sqrt{p})/\mathbf{Q}_p$ is a quadratic extension.

Question 2 (2 points)

Show that $\mathbf{Q}_p$ would not even be an integral domain if p is not a prime number. (`Hint:` Take $p = 6$ and look for an element $\alpha \in \mathbf{Q}_6$ with $\alpha(\alpha+1) = 0$, either using Hensel's lemma, or by an inductive construction.
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Question 3 (10 points)

Consider the quadratic extension $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$ with the corresponding ring extension $\mathbf{Z}[\sqrt{-1}]/\mathbf{Z}$. For an element $\alpha = a + b\sqrt{-1} \in \mathbf{Q}(\sqrt{-1})$, define $N(\alpha) = a^2 + b^2$.

(a) (2 points) For any element $\alpha, \beta \in \mathbf{Q}(\sqrt{-1})$, show that $N(\alpha\,\beta) = N(\alpha)\,N(\beta)$.

(b) (2 points) Determine all the units of $\mathbf{Z}[\sqrt{-1}]$ using previous part.

(c) (2 points) Use previous part to find the prime factorization of 2 in $\mathbf{Z}[\sqrt{-1}]$. (Hint: $\mathbf{Z}[\sqrt{-1}]$ is a UFD.)

(d) (2 points) Use this information to find the ramification index of ν_2 (as a valuation on $\mathbf{Q}$) in $\mathbf{Q}(\sqrt{-1})$.

(e) (2 points) Let $\nu\colon \mathbf{Q}(\sqrt{-1}) \to \mathbf{Z} \cup \{\infty\}$ denote the extension of ν_2. Find $\nu(12)$.