# MATH 201
# ÉNONCÉS DES EXERCICES 4

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(1) Décidez si les ensembles suivants sont ouvertes, fermés, connexes, simplement connexes et pour chaqun déterminez la decomposition $\mathrm{int}(S) \sqcup \partial S \sqcup \mathrm{ext}(S)$ de $\mathbf{R}^n$.
- $S = \mathbf{Q} \subseteq \mathbf{R}$,
- $S = \{x \in \mathbf{Q} \,|\, 0 < x < 1\} \subseteq \mathbf{R}$,
- $S = \{1/n \in \mathbf{R} \,|\, n \in \mathbf{N}\} \subseteq \mathbf{R}$,
- $S = \{1/n \in \mathbf{R} \,|\, n \in \mathbf{N}\} \cup \{0\} \subseteq \mathbf{R}$,
- $S = \{(-1)^n + 1/n \in \mathbf{R} \,|\, n \in \mathbf{N}\} \subseteq \mathbf{R}$,
- $S = \{(-1)^n + 1/n \in \mathbf{R} \,|\, n \in \mathbf{N}\} \cup \{0\} \subseteq \mathbf{R}$,
- $S = \{(x,y) \in \mathbf{R}^2 \,|\, 1 \leq d((x,y),0) < 2\} \subseteq \mathbf{R}^2$,
- $S = \{(x,y) \in \mathbf{Q}^2 \,|\, 0 \leq x, y \leq 1\} \subseteq \mathbf{R}^2$,
- $S = \{(x,y) \in \mathbf{Q}^2 \,|\, 0 \leq x, y < 1\} \subseteq \mathbf{R}^2$,
- $S = \{(x,y) \in \mathbf{Q}^2 \,|\, 0 < x, y \leq 1\} \subseteq \mathbf{R}^2$,
- $S = \{(x,y) \in \mathbf{Q}^2 \,|\, 0 < x, y < 1\} \subseteq \mathbf{R}^2$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z = 0, \text{ et } x^2 + y^2 \leq 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z = 0, \text{ et } x^2 + y^2 < 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z = 0, \text{ et } x^2 + y^2 \geq 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z = 0, \text{ et } x^2 + y^2 > 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z \in [0,1], \text{ et } x^2 + y^2 \leq 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z \in [0,1], \text{ et } x^2 + y^2 < 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z \in [0,1], \text{ et } x^2 + y^2 \geq 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, z \in [0,1], \text{ et } x^2 + y^2 > 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, x = y \text{ et } x = z\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, x = y \text{ ou } x = z\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, x^2 + y^2 = 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, x^2 + y^2 \leq 1\} \subseteq \mathbf{R}^3$,
- $S = \{(x,y,z) \in \mathbf{R}^3 \,|\, x^2 + y^2 < 1\} \subseteq \mathbf{R}^3$,

(2) Dessinez le graphe des fonctions suivantes en déterminant quelques courbes de niveau :
- $f(x,y) = x + y$
- $f(x,y) = x$
- $f(x,y) = y$
- $f(x,y) = 2x - 3y$
- $f(x,y) = x^2$
- $f(x,y) = 1 - y^2$
- $f(x,y) = y^3$
- $f(x,y) = e^x$
- $f(x,y) = \sin(y)$
- $f(x,y) = 1 - (x^2 + y^2)$
- $f(x,y) = 1 - \sqrt{x^2 + y^2}$

(3) Calculez les limites suivantes s'ils existent. Sinon, montrer que la limite n'existe pas :
- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy\cos(y)}{3x^2 + y^2}$
- $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$
- $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

- $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2(y)}{x^2+2y^2}$
- $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$
- $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$
- $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

(4) Déterminez tout les points $X \in \mathbf{R}^2$ pour lequels la fonction :
- $f(x,y) = \cos(\sqrt{1-x+y})$
- $f(x,y) = \frac{e^x+e^y}{e^{xy}-1}$
- $f(x,y) = \frac{1+x^2+y^2}{1-x^2-y^2}$

est continue.

(5) Est-ce que la fonction

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

continue en $(0,0)$?

(6) Calculez les derives partielles suivantes :
- $\frac{\partial}{\partial x}\left(1-x^2-2y^2\right)$
- $\frac{\partial}{\partial y}\left(1-x^2-2y^2\right)$
- $\frac{\partial}{\partial x}\left(x^2+x^2y^3+y^3\right)$
- $\frac{\partial}{\partial y}\left(x^2+x^2y^3+y^3\right)$
- $\frac{\partial}{\partial x}\left(\frac{x}{y}\right)$
- $\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$
- $\frac{\partial}{\partial x}\left(\frac{x}{(x+y)^2}\right)$
- $\frac{\partial}{\partial y}\left(\frac{x}{(x+y)^2}\right)$
- $\frac{\partial}{\partial x}\left(\ln(x+2y+3z)\right)$
- $\frac{\partial}{\partial y}\left(\ln(x+2y+3z)\right)$
- $\frac{\partial}{\partial z}\left(\ln(x+2y+3z)\right)$
- $\frac{\partial}{\partial x_1}\left(x_1^2x_2+e^{x_1x_2+x_3x_4}\right)$
- $\frac{\partial}{\partial x_2}\left(x_1^2x_2+e^{x_1x_2+x_3x_4}\right)$
- $\frac{\partial}{\partial x_3}\left(x_1^2x_2+e^{x_1x_2+x_3x_4}\right)$
- $\frac{\partial}{\partial x_4}\left(x_1^2x_2+e^{x_1x_2+x_3x_4}\right)$

(7) Soient $f, g\colon S \to \mathbf{R}$ deux fonctions, $X_\circ \in S$ un point quelconque; où $S \subseteq \mathbf{R}^n$ une partie ouverte et connexe. Montrez que si

$$\frac{\partial}{\partial x_i} f\,|_{X_\circ} \text{ et } \frac{\partial}{\partial x_i} g\,|_{X_\circ}$$

existent pour tout $i \in \{1, 2, \ldots, n\}$, alors pour tout $i = 1, 2, \ldots, n$ on a:

- $\frac{\partial}{\partial x_i}(f + g) \mid_{X_o} = \frac{\partial}{\partial x_i} f \mid_{X_o} + \frac{\partial}{\partial x_i} g \mid_{X_o}$
- $\frac{\partial}{\partial x_i}(fg) \mid_{X_o} = \left(\frac{\partial}{\partial x_i} f \mid_{X_o}\right) g(X_o) + f(X_o) \left(\frac{\partial}{\partial x_i} g \mid_{X_o}\right)$
- $\frac{\partial}{\partial x_i}\left(\frac{f}{g}\right) \mid_{X_o} = \frac{\left(\frac{\partial}{\partial x_i} f|_{X_o}\right) g(X_o) - f(X_o)\left(\frac{\partial}{\partial x_i} g|_{X_o}\right)}{(g(X_o))^2}$