MATH 504
EXERCISES 1

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Unless otherwise stated G is a group.

(1) Let G be a semi-group. Show that G is a group if and only if the following conditions hold :
   - there exists an element $e \in G$ such that $ea = a$ for all $a \in G$ (i.e. left identity)
   - for each $a \in G$, there exists an element $a^{-1} \in G$ such that $a^{-1}a = e$ (i.e. left inverse).

   Deduce that a semi-group G is a group if and only if for all $a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions in G.

(2) Let $(G, +)$ be a group, S a non-empty set. By $F(G, S)$ we denote the set of all functions $G \to S$. On $F(G, S)$ define $(f * f')(x) = f(x) + f'(x)$ (i.e. pointwise addition).
   - Prove that $F(G, S)$ is a group.
   - Prove that $F(G, S)$ is abelian if G is abelian.

(3) Show that all subgroups of abelian groups are normal.

(4) Let $G = (\mathbf{Z}, +)$.
   - Show that if H is a subgroup of G then there is some integer n so that $H = n\mathbf{Z} = \{kn \mid k \in \mathbf{Z}\}$.
   - Determine all subgroups of $\mathbf{Z}/p\mathbf{Z}$; where p is a fixed prime number.
   - Determine all subgroups of $\mathbf{Z}/n\mathbf{Z}$.

(5) Let G, G' be two groups. On the set $G \times G'$, define $(g_1, g_1') * (g_2, g_2') := (g_1 g_2, g_1' g_2')$.
   - Show that $(G \times G', *)$ is a group.
   - Show that $(G \times G')$ is abelian if and only if both G and G' are abelian.
   - Write out an addition table for the following groups :
     - $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ (this is called the Klein 4 group)
     - $\mathbf{Z}/2\mathbf{Z} \times \mathfrak{S}_3$.

(6) Write out the multiplication table of a group of order 2 (or 3). Deduce that G is abelian.

(7) Fix a prime number p. We let :
$$R_p = \left\{ \frac{a}{b} \in \mathbf{Q} \mid \gcd(a, b) = 1, \text{ and } \gcd(b, p) = 1 \right\},$$
$$R^p = \left\{ \frac{a}{b} \in \mathbf{Q} \mid \gcd(a, b) = 1, \text{ and } b = p^k \text{ for some } k \in \mathbf{N} \cup \{0\} \right\}$$
   Show that both R_p and R^p are groups under addition.

(8) Let G be a group. Show that the following conditions on G are equivalent :
   - G is abelian,
   - $(gg')^2 = g^2(g')^2$ for any $g, g' \in G$
   - $(gg')^{-1} = g^{-1}(g')^{-1}$ for any $g, g' \in G$
   - $(gg')^n = g^n(g')^n$ for any $g, g' \in G$ and for all $n \in \mathbf{Z}$
   - $(gg')^n = g^n(g')^n$ for 3 consequtive positive integers $n = k, k+1, k+2$.

   Deduce that a group G is abelian if an only if the map
$$\begin{aligned} \mathrm{inv}\colon G &\to G \\ g &\mapsto \mathrm{inv}(g) := g^{-1} \end{aligned}$$
   is an automorphism.

(9) Let $\varphi\colon G \to G'$ be a group homomorphism. Show that φ is an isomorphism if and only if there is a group homomorphism $\psi\colon G \to G'$ so that both $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$.

(10)
   - For H and K two subgroups of G show that $H \cap K$ is also a subgroup of G.

- Let $\mathcal{A}$ be a non-empty collection of subgroups of G. Show that
$$\bigcap_{H\in\mathcal{A}} H$$
is a subgroup of G.
- Let A be any *subset* of G. The subgroup generated by A, denoted $\langle A\rangle$, is defined as :
$$\langle A\rangle := \bigcap_{H\leq G,\, A\subseteq H} H.$$
Deduce, using previous part of the exercise that $\langle A\rangle$ is a subgroup of G.
- Let B be any *subset* of G. Define
$$\widetilde{B} := \{a_1^{\alpha_1} a_2^{\alpha_2}\cdots a_k^{\alpha_k} | n\in\mathbf{N},\ \alpha_i = \pm 1 \text{ with } a_i \text{ not necessarily distinct}\}.$$
Show that $\widetilde{B}$ is a subgroup of G.
- Show finally that for any subset A of a group G, we have $\langle A\rangle = \widetilde{A}$.
- Show that if $A\subseteq B$ then $\langle A\rangle \subseteq \langle B\rangle$.
- Show that if $H\leq G$ then $\langle H\rangle = H$.
- When $A = \{g_1,\dots g_n\}$ is a finite subset of G, we denote $\langle A\rangle$ by $\langle g_1,\dots,g_n\rangle$. In particular, when $A=\{g\}$. we denote $\langle A\rangle = \langle g\rangle$. In $\mathfrak{S}_4$, determine $\langle\{(1\,2),(2\,3\,4)\}\rangle$
- Compute $\langle(1\,2\,3)\rangle$ in $\mathfrak{S}_4$.
- Compute $\langle(1\,2),(1\,3),(1\,4)\rangle$ in $\mathfrak{S}_4$.
- Compute $\langle(1\,2),(1\,3),(1\,4)\rangle$ in $\mathfrak{S}_5$.
- Show that $H\cup K$ is in general not a subgroup of G.

(11) Let H and K be two subgroups of a group G. Show that if G is abelian, then
$$HK := \{hk\,|\, h\in H,\ kinK\}$$
is a subgroup of G. Give an example to show that assuming G to be an abelian group is necessary.

(12) For an arbitrary group G, a subgroup H of G is called *cyclic* if there is some $h\in H$ so that $H=\langle h\rangle$. In particular, we say that a group G is cyclic if $G=\langle g\rangle$ for some $g\in G$.
- Show that $\mathbf{Z}$ is cyclic, however its generator $g\in G$ is not unique.
- Show that if G is cyclic and H is a subgroup of G then H is cyclic, too.
- Using previous exercise, determine all subgroups of $\mathbf{Z}$.
- Show that if a cyclic group has only one generator, then it is of order 2.
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(13) Let G be a group of $g_o\in G$ be a fixed element.
- Show that the set $C_G(g_o) := \{g\in G\,|\, g\,g_o = g_o\,g\}$ is a subgroup of G. This subgroup is called the *centralizer* of g_o in G.
- For $G=\mathfrak{S}_4$, compute $C_G(\sigma)$; where $\sigma_o = (1\,2), (1\,2\,3), (1\,2\,3\,4)$.
- Show that if G is abelian, then for any $g_o\in G$, $C_G(g_o) = G$.
- More generally, let X be any non-empty subset of G and define
$$C_G(X) := \{g\in G\,|\, gx = xg \text{ for all } x\in X\}.$$
Show that $C_G(X)$ is a subgroup of G.
- Show that $C_G(X) = C_G(\langle X\rangle)$; where $\langle X\rangle$ denotes the subgroup of G generated by X, (see Exercise 10).
- Deduce that $C_G = Z_G = C_G(G) = \bigcap_{g\in G} C_G(g)$ is a subgroup of G called the center of G.

(14) Let G be a group and X be a non-empty subset of G.
- Show that the set
$$N_G(X) = \{g\in G\,|\, gxg^{-1}\in X\}$$
is a subgroup of G, called the normalizer of X in G.
- Show that for any X $C_G(X)\leq N_G(X)$.
- Show that $N_G(X) = N_G(\langle X\rangle)$.
- Show that X is a normal subgroup of G if and only if $N_G(X) = G$.
- Deduce that $Z(G)$ is always a normal subgroup of G.
- For $X = \{(1\,2),(1\,3),(2\,3)\}\subset\mathfrak{S}_4$ compute $N_G(X)$.

(15) Let G be a group and $g \in G$ be an arbitrary element. We define the *order* of g, denoted by $\operatorname{ord} g$, to be the size of the subgroup generated by g, that is $\operatorname{ord} g = |\langle g \rangle|$ (infinity is allowed!).
- ▶ Find the order of $(1\,2\,4\,3\,5\,6\,8\,7) \in \mathfrak{S}_8$.
- ▶ Find the order of $(1\,2\,4\,3)(5\,6\,8\,7) \in \mathfrak{S}_8$.
- ▶ Find the order of $(1\,2\,4)(\,3\,5\,6\,8\,7) \in \mathfrak{S}_8$.
- ▶ In $\mathfrak{S}_5$ determine all elements of order 4.
- ▶ For an element $x \in G$ show that x and x^{-1} have the same orders.
- ▶ Prove that for any $x, g \in G$ we have $\operatorname{ord} xgx^{-1} = \operatorname{ord} g$.
- ▶ Show, using the previous part, that for any $x, y \in G$ we have $\operatorname{ord} xy = \operatorname{ord} yx$.

(16) Show that if $\varphi\colon G \to G'$ is a group homomorphism with $G = \langle g_o \rangle$, then show that φ is determined only by $\varphi(g_o)$.

(17) Let G be a group, g and g' be fixed elements of G. Define

$$\begin{aligned} \varphi_{g,g'}\colon \mathbf{Z} \times \mathbf{Z} &\to G \\ (k, l) &\mapsto g^k(g'^l) \end{aligned}$$

Determine condition(s) on the elements g and g' so that the map $\varphi_{g,g'}$ is a group homomorphism.

(18) Prove that the following groups are not isomorphic :
- ▶ $\mathbf{R}$ and $\mathbf{Q}$
- ▶ $\mathbf{Q}$ and $\mathbf{Z}$

(19) Let G be a group and H be a subgroup of G. Show that the following conditions on H are equivalent :
  i. $ghg^{-1} \in H$ for all $g \in G$, for all $h \in H$
  ii. $gHg^{-1} = H$ for all $g \in G$
  iii. $gH = Hg$ for all $g \in G$.

(20) Let G be a group and let $\operatorname{Aut}(G)$ denote the set of all automorphisms of G.
- ▶ Show that $\operatorname{Aut}(G)$ is a group under composition.
- ▶ For a fixed $g_o \in G$ show that the map :

$$\begin{aligned} \varphi_{g_o}\colon G &\to G \\ x &\mapsto g_o x g_o^{-1} \end{aligned}$$

  is an element of $\operatorname{Aut}(G)$. Such an automorphism is called an inner automorphism of G.
- ▶ Show that the map :

$$\begin{aligned} \iota\colon G &\to \operatorname{Aut}(G) \\ g_o &\mapsto \varphi_{g_o} \end{aligned}$$

  is a monomorphism.
- ▶ Show that the image of ι is a normal subgroup of $\operatorname{Aut}(G)$. This subgroup denote by $\operatorname{Inn}(G)$ and the quotient $\operatorname{Aut}(G)/\operatorname{Inn}(G)$ is called the group of outer automorphisms of G and denoted by $\operatorname{Out}(G)$.

(21) Let G, G' be two groups and $\varphi\colon G \to G'$ be a group homomorphism. Show that if H and H' are normal subgroup of G and G' respectively, then φ induces a group homomorphism $\widehat{\varphi}\colon G/H \to G'/H'$ when $\varphi(H) \subseteq H'$.

(22) Let G be a group, H and K are subgroups of G so that $H \leq N_G(K)$.
- ▶ Show that $HK := \{hk \mid h \in H,\ k \in K\}$ is a subgroup of G.
- ▶ Show that K is a normal subgroup of HK.
- ▶ Show that $H \cap K$ is a normal subgroup of H.
- ▶ Show that $HK/K \cong H/H \cap K$. Hint: Use 1st isomorphism theorem.
- ▶ This is called diamond isomorphism theorem. Explain why?

(23) Let G be a group and H be a subgroup of G.
- ▶ Show that for any $g \in G$ the set gHg^{-1} is a subgroup of G.
- ▶ Show that if G has only one subgroup of order $|H|$; then H is a normal subgroup.