MATH 504
EXERCISES 2

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Unless otherwise stated G is a group.

(1) Consider the following map :

$$\mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$$
$$(r,(x,y)) \mapsto r \cdot (x,y) := (x+r, y-r)$$

- ▶ Show that this defines an action of $(\mathbf{R},+)$ on $\mathbf{R}^2$.
- ▶ Sketch the orbits of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.
- ▶ Determine the stabilizers of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.

(2) Consider the following map :

$$\mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$$
$$(r,(x,y)) \mapsto r \cdot (x,y) := \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- ▶ Show that this defines an action of $(\mathbf{R},+)$ on $\mathbf{R}^2$.
- ▶ Sketch the orbits of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.
- ▶ Determine the stabilizers of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.
- ▶ Determine all $a, b \in \mathbf{R}$ so that the following twisted version of the above map still defines an action of $\mathbf{R}$ on $\mathbf{R}^2$

$$\mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$$
$$(r,(x,y)) \mapsto r \cdot (x,y) := \begin{pmatrix} a\cos(r) & -b\sin(r) \\ b\sin(r) & a\cos(r) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(3) Consider the following map :

$$\mathbf{R} \setminus \{0\} \times \mathbf{R}^2 \to \mathbf{R}^2$$
$$(r,(x,y)) \mapsto r \cdot (x,y) := (rx, ry)$$

- ▶ Show that this defines an action of $(\mathbf{R} \setminus \{0\}, \times)$ on $\mathbf{R}^2$.
- ▶ Sketch the orbits of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.
- ▶ Determine the stabilizers of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.

(4) Consider the following map :

$$\mathbf{R} \setminus \{0\} \times \mathbf{R}^2 \to \mathbf{R}^2$$
$$(r,(x,y)) \mapsto r \cdot (x,y) := (rx, r^{-1}y)$$

- ▶ Show that this defines an action of $(\mathbf{R} \setminus \{0\}, \times)$ on $\mathbf{R}^2$.
- ▶ Sketch the orbits of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.
- ▶ Determine the stabilizers of the points $(0,0),(1,0),(0,1) \in \mathbf{R}^2$.

(5) Let G be a finite p group acting on a X, where $|X| = k < \infty$ with $p \nmid k$. Prove that the group G has a fixed point, i.e. there is some $x \in X$ so that $g \cdot x = x$ for all $g \in G$. Deduce that the center of such a group G cannot be trivial.

(6) Let G be a group acting on a set X. For some $g \in G$ we define :

$$\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}.$$

- ▶ Consider the action of $\mathfrak{S}_n$ on $\{1,2,\ldots,n\}$. For an element $\sigma \in \mathfrak{S}_n$ compute $\mathrm{Fix}(\sigma)$

- Let H be any subgroup of G and consider the action of H on G by multiplication from left :

$$H \times G \to G$$
$$(h, x) \mapsto h \cdot x := hx$$

For $e \in H$, find $\mathrm{Fix}(e)$.
- Suppose that the action of G on X is *transitive*, i.e. for any $x, y \in X$, there is an element $g \in G$ so that $g \cdot x = y$. Show that :

$$\frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g) = 1.$$

Remark that the sum on the left hand side represents an average over the group G!

(7) Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group[1] of order 8.
- Show that Q_8 is isomorphic to a subgroup of $\mathfrak{S}_8$.
- Show that Q_8 cannot be isomorphic to any subgroup of $\mathfrak{S}_n$ for $n \leq 7$. Hint: Note first that such an action would induce an action of Q_8 on a set, say X, with $n \leq 7$ elements. Show that if this is the case, then for and $x \in X$, $\mathrm{Stab}(x)$ must contain the subgroup $\{\pm 1\}$.

(8) Write out the class equation for :
- $G = \mathfrak{S}_5$
- $G = Q_8$

(9) Let $g_1, \ldots, g_r$ are representatives of distinct conjugacy classes of a group G. Show that if these elements commute, i.e. $g_i g_j = g_j g_i$ for any $i, j \in \{1, 2, \ldots, r\}$, then G is an abelian group.

(10) Let G be a group and H be a normal abelian subgroup of G.
- Show that G/H acts on H by conjugation.
- Deduce that one obtains a homomorphism from G/H to $\mathrm{Aut}(H)$.

(11) Consider the action of $G = \mathbf{Z}/4\mathbf{Z}$ onto itself from left :

$$G \times G \to G$$
$$(g, x) \mapsto g \cdot x = g + x$$

Describe explicitly the permutation representation obtained from this action.

(12) Let $X = \{(i, j) \mid i, j \in \{1, 2, 3, 4\}\}$.Consider :

$$\mathfrak{S}_4 \times X \to X$$
$$(\sigma, (i, j)) \mapsto \sigma \cdot (i, j) := (\sigma(i), \sigma(j))$$

- Show that the above map defines a group action.
- Determine the orbit of $(1, 1)$ and $(1, 2)$.
- Compute the images of the elements $\sigma_1 = (1\,2)$, $\sigma_2 = (1\,2\,3)$, $\sigma_3 = (1\,2\,3\,4)$ and $\sigma_4 = (1\,2)(3\,4)$ under the permutation representation associated to this action i.e. $\pi \colon \mathfrak{S}_4 \to \mathfrak{S}_{16}$

[1] This is a group under multiplication : $ii = jj = kk = -1$, $ij = k$, $jk = i$, $ki = j$ and $ji = -k$, $kj = -i$, $ik = -j$.