MATH 504
EXERCISES 2

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Unless otherwise stated G is a group.

(1) Show that the group $\mathbf{Z}/2\mathbf{Z}$ is the only group with exactly 2 conjugacy classes.

(2) Let G be a group and H, K be two subgroups of G.
- ▶ Show, by an example that, their product HK may not be a subgroup of G.
- ▶ Show that HK is a subgroup if and only if $HK = KH$.
- ▶ Deduce that HK is a subgroup whenever either H or K (or both) is a normal subgroup of H.
- ▶ Show that if H and K are normal and disjoint subgroups of G, then fpr any $h \in H$ and any $k \in K$ $hk = kh$.
- ▶ Deduce that any group of order 45 is cyclic.

(3) Let N be a normal subgroup of G. Show that if both N and G/N are p groups then G is a p group.

(4) Show that if G is a finite p group and H is a non-trivial normal subgroup of G, then $H \cap Z(G) \neq \{e\}$.

(5) Show that if G is a finite group and p_i, where $i = 1, 2, \ldots, k$ is the list of all distinct prime numbers that divide $|G|$. Show that if $n_{p_i} = 1$ for all $i = 1, 2, \ldots, k$; then

$$G \cong P_1 \times P_2 \times \ldots \times P_k;$$

where $\{P_i\} = \mathrm{Syl}_{p_i}(G)$.

(6) Find Sylow 2 and 3 subgroups of $\mathfrak{S}_n$ where $n = 3, 4$.

(7) Show that if N is a normal subgroup of G and in P is a Sylow p subgroup of G lying inside N, then P is normal in G, hence the unique Sylow p subgroup of G.

(8) Let G be a group of order 105.
- ▶ Show that either $n_5 = 1$ or $n_7 = 1$.
- ▶ Show that in either case G has a subgroup of order 35, denoted by H.
- ▶ Show that H is normal in G.
- ▶ Show that H is cyclic.
- ▶ Using Exercise 7 show that both the Sylow 5 subgroup and the Sylow 7 subgroup is normal in G.

(9) Prove that groups of order :
- ▶ 56
- ▶ 63
- ▶ 96
- ▶ 255
- ▶ 312
- ▶ 351
- ▶ 462

are not simple.

(10) Determine the number of elements of order 7 in a group G of order 168.

(11) Let p and q be distinct prime numbers and G be a group of order p^2q. Prove that G is not simple. Hint: Investigate the cases $p > q$ and $p < q$ separately.

(12) Let G be a finite group and let p be the smallest prime number dividing $|G|$. Suppose that $P \in \mathrm{Syl}_p(G)$ is cyclic. Show that $N_G(P) = C_G(P)$.

(13) Let p be an arbitrary odd prime number and $G = \mathfrak{S}_{2p}$.
- ▶ Determine an element, say P, of $\mathrm{Syl}_p(G)$.

- ▶ Show that P is an abelian group. Hint: What is the order of P?

(14) Let P be an arbitrary Sylow p subgroup of G. Show that $N_G(N_G(P)) = N_G(P)$.

(15) Prove that a group of order 91 is cyclic.