MATH 504
EXERCISES 5

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(1) Let R be an integral domain. We say that a subset S of R is multiplicative if $0 \notin S$ and whenever $x, y \in S$, $xy \in S$, too. Throughout this exercise assume S is a multiplicative subset of R.
- Show that the relation $(p, q) \sim (r, s) :\Leftrightarrow ps - qr = 0$ is an equivalence relation on $R \times S$.
- We let $S^{-1}R$ denote the set of equivalence classes and write p/q for the equivalence class of (p, q). On this set, we define
$$p/q + r/s = (ps + qr)/(qs) \text{ and } p/q \cdot r/s = (pr)/(qs).$$
Show that these binary operations are well-defined.
- Determine the additive and multiplicative identities.
- Show that both operations are commutative.
- Show that $S^{-1}R$ is an integral domain, too.
- Show that $S = \{1\}$ is a multiplicative subset of R and determine $S^{-1}R$ in this case.
- Show that the map :
$$\begin{aligned} \varphi\colon R &\to S^{-1}R \\ r &\mapsto r/1 \end{aligned}$$
is an injective ring homomorphism.
- Show that for any ideal I of R, the set $I^{ext} := \{a/s : a \in I, s \in S\}$ is an ideal of $S^{-1}R$.
- Show that for any ideal J of $S^{-1}R$, the set $I^{contr} := \varphi^{-1}(J)$ is an ideal of R.
- Show that the above maps induce a one to one correspondence between prime ideals of $S^{-1}R$ and prime ideals of R which do not intersect with S.
- Show that if $\mathfrak{p}$ is a prime ideal of R, then the set $R \setminus S$ is a multiplicative subset.
- Deduce that whenever $S = R \setminus \mathfrak{p}$, then this gives a one to one correspondence between prime ideals of $S^{-1}R$ and prime ideal that lie in $\mathfrak{p}$.
- Deduce that if $S = R \setminus \mathfrak{p}$, then the ideal $\mathfrak{p}^{ext}$ is the unique maximal ideal of $S^{-1}R$. (Rings with a unique maximal ideal are called *local rings*.)
- Show that $\{0\}$ is a prime ideal of R, and hence this construction generalizes the construction of field of fractions that we discussed in class. Verify that in this case $S^{-1}R$ is a field.

(2) Let $R = \mathbf{Z}[\sqrt{-1}]$
- Show that the ring $R/(1 + i)$ is a field. Determine this field.
- Let $q \in \mathbf{Z}$ with $q \equiv 3 \pmod 4$. Prove that $R/\langle q \rangle$ is a field. with q^2-many elements.

(3) Let $R = \mathbf{Z}[\sqrt{-5}]$ and consider the ideals $I = \langle 2, 1 + \sqrt{-5} \rangle = \langle 2, 1 - \sqrt{-5}$, $J = \langle 3, 2 + \sqrt{-5} \rangle$ and $J' = \langle 3, 2 - \sqrt{-5} \rangle$.
- Determine which of the generators of the above ideals are irreducible.
- Show that no two of the above generators are associates.
- Prove that the ideals I, J and J' are all prime.

(4) Let $R = \mathbf{Z}[X, Y]$.
- Show that the ideals $I = \langle x, y \rangle$ and $J = \langle p, X, Y \rangle$ for any prime number p are both prime ideals.
- Show that I is not maximal, but J is.

(5) Let k be any field. Show that the two rings $k[X, Y]/\langle X - Y^2 \rangle$ and $k[X, Y]/\langle X^2 - Y^2 \rangle$ cannot be isomorphic.

(6) Determine all ideals of the ring $\mathbf{Z}[X]/\langle 2, X^2 + 1 \rangle$

(7) Let R be any ring. We say that the finite basis condition (FBC) for ideals hold in R if for each ideal I of R, there is a finite set, say $r_1, \dots, r_n$, of elements of I so that $I = \langle r_1, r_2, \dots, r_n \rangle = I$.
- Recall the definition of ascending chain condition (ACC).
- Show that (ACC) is equivalent to (FBC).