Université Galatasaray, Département de Mathématiques
2018 - Fall Semester – Math 504 - Advanced Algebra
Final Exam, 12 December 2018 – Ayberk Zeytin, 120 minutes
Name & Surname: ____ **Sign:** _______________

Question:	1	2	3	4	Total
Points:	2	18	8	12	40
Score:					

Question 1 (2 points)
Prove that a group G of order 72 cannot be simple. (`Hint:` Consider the Sylow 3 subgroups of G.)

Question 2 (18 points)

Let k be an arbitrary field.

(a) (2 points) Let p and q be non-negative integers. Show that the groups $\mathbf{Z}/n\mathbf{Z}$ and $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/q\mathbf{Z}$ are isomorphic if and only if $n = pq$ and p and q are relatively prime.

(b) (2 points) Deduce that if $m = d_1 \cdot \ldots \cdot d_l$ with integers $d_1, \ldots, d_l$ being pairwise relatively prime, then we have $\mathbf{Z}/m\mathbf{Z} \cong \mathbf{Z}/d_1\mathbf{Z} \times \ldots \times \mathbf{Z}/d_l\mathbf{Z}$. (`Hint:` Use part (a) and induction.)

(c) (2 points) Show that a non-zero polynomial $\mathsf{f}(\mathsf{X}) \in \mathsf{k}[\mathsf{X}]$ can have at most $\deg(\mathsf{f})$-many zeros in k.

(d) (2 points) Show that if G is a finite subgroup of the multiplicative group $\mathsf{k}^\times$ then G is cyclic. (`Hint:` Say G is of order m. Find a polynomial which admits every element of G as a root. Use the classification theorem of finite abelian groups and parts (b) and (c) to conclude.)

(e) (2 points) Deduce that the multiplicative group of a finite field is cyclic.

(f) (2 points) Show that the group $\mathbf{Z} \times \mathbf{Z}$ is not cyclic.

(g) (2 points) Show that $\mathbf{Q}$ and $\mathbf{Q} \times \mathbf{Q}$ cannot be isomorphic as additive groups using parts (d) and (f).

(h) (2 points) Construct a field of size 8, say K.

(i) (2 points) Find a generator of the multiplicative group $\mathsf{K}^\times$, say α. Write the elements $\alpha^2, \alpha^4, \alpha^6$ in terms of polynomials. What can you say about the multiplication table of the group $\mathsf{K}^\times$.

Question 3 (8 points)

Let R be any commutative ring with identity.

(a) (2 points) Show that if R is a PID and I is an ideal of R, then R/I is also a PID.

(b) (2 points) Show that every prime ideal of R is maximal if R is a PID.

(c) (2 points) Let $\mathsf{R} = \mathsf{k}[\mathsf{X}]$; where k is a field (so that it is a PID). Show that there is a one to one correspondence between maximal ideals of R and monic irreducible polynomials in R.

(d) (2 points) True/False : if $\mathsf{r} \in \mathsf{R}$ is irreducible (R is just a commutative ring with identity), then $\langle \mathsf{r} \rangle$ is maximal.

Question 4 (12 points)

Decide whether the following statements are true or false and circle your claim. If true give a proof, if false give a counter-example

(a) (2 points) Let G be a group and g_1, g_2 be two elements of orders n and m in G. Then the order of the product $g_1 g_2$ divides nm.

True / False :

(b) (2 points) The groups $(\mathbf{Q}, +)$ and $(\mathbf{Q} \setminus \{0\}, \cdot)$ are not isomorphic.

True / False :

(c) (2 points) If $\varphi \colon G \to G'$ is an isomorphism of groups, then its inverse (as a bijection from G' to G) is a group homomorphism.

True / False :

(d) (2 points) If R is an integral domain and I is an ideal of R, then R/I is an integral domain, too.

True / False :

(e) (2 points) The extension $\mathbf{Q}(\sqrt[3]{\pi})/\mathbf{Q}$ is algebraic.

True / False :

(f) (2 points) The extension $\mathbf{Q}(\sqrt[3]{\pi})/\mathbf{Q}(\pi)$ is algebraic.

True / False :