Université Galatasaray, Département de Mathématiques
2018 - Fall Semester – Math 504 - Advanced Algebra
Mid Term Exam, 21 November 2018 – Ayberk Zeytin, 120 minutes
Name & Surname: ______________________ **Sign:** __________

Question:	1	2	3	Total
Points:	2	18	6	26
Score:				

Question 1 (2 points)
Let p and q be two distinct prime numbers and G is a group of order p^2q. Assume that $q \nmid p^2 - 1$ and $p \nmid q - 1$. Show that G is abelian.

Question 2 (18 points)

Let G be a group. For arbitrary elements $x, y \in G$, the commutator of x, y is defined as

$$[x, y] := xyx^{-1}y^{-1}.$$

(a) (2 points) We let $[G, G]$ be the subgroup of G generated by all the commutators in G. Give a necessary and sufficient condition so that $[G, G] = \{e\}$.

(b) (2 points) Let G, H be two groups and $\varphi \colon G \to H$ be a group homomorphism. Show that φ induces a group homomorphism from $[G, G]$ to $[H, H]$.

(c) (2 points) Show that if the group homomorphism φ of part (b) is surjective, then so is the induced homomorphism from $[G, G]$ to $[H, H]$.

(d) (2 points) Show that $[\mathsf{G}, \mathsf{G}]$ is a normal subgroup of G. (`Hint:` You may use part (c).)

(e) (2 points) Show that $\mathsf{G}/[\mathsf{G}, \mathsf{G}]$ is commutative.

(f) (2 points) Show that if G' is a subgroup of G which contains $[\mathsf{G}, \mathsf{G}]$ then G' is a normal subgroup.

(g) (2 points) Fix an integer $n \geq 4$ and let $\sigma, \tau \in \mathfrak{S}_n$ be two transpositions. Show that $\sigma\tau$ is the product of two commutators. (`Hint:` Investigate the cases where σ and τ being disjoint and non-disjoint separately.)

(h) (2 points) Deduce that an odd permutation cannot be an element of $[\mathfrak{S}_n, \mathfrak{S}_n]$. (`Hint:` Recall the definition of the sign of a permutation, and that sign defines a homomorphism from $\mathfrak{S}_n$ to $\{\pm 1\}$.)

(i) (2 points) For $G = \mathfrak{S}_n$, determine $[G, G]$.

Question 3 (6 points)

Let G be a group of order 132. In this exercise, we will show that G cannot be simple. So suppose that G is simple.

(a) (2 points) Determine the number of elements of order 11 in G.

(b) (2 points) Determine the number of elements of order 3 in G.

(c) (2 points) Show that G must have a unique Sylow 2 subgroup, and conclude.