Université Galatasaray, Département de Mathématiques
2018 - Fall Semester – Math 504 - Advanced Algebra
Mid Term Exam, 12 December 2018 – Ayberk Zeytin, 120 minutes
Name & Surname: ______________________ **Sign:** ____________

Question:	1	2	Total
Points:	16	8	24
Score:			

Question 1 (16 points)
Let R be a commutative ring with identity. An element $r \in R$ is called **nilpotent** if $r^n = 0$ for some positive integer n.

(a) (2 points) Determine all nilpotent elements in $\mathbf{Z}/12\mathbf{Z}$.

(b) (2 points) Show that if $r \in R$ is a nilpotent element then $1 + r$ is a unit.

(c) (2 points) Deduce that the sum of a nilpotent element and a unit is a unit in R.

(d) (2 points) Deduce that an element $f(X) = a_0 + a_1 X + \ldots + a_n X^n \in R[X]$ is a unit if a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent elements of R.

(e) (2 points) Give an example of a nilpotent and a unit element in $(\mathbf{Z}/8\mathbf{Z})[\mathsf{X}]$ of degree ≥ 1.

(f) (2 points) Show that the set $\mathfrak{n}(\mathsf{R})$ of all nilpotent elements in R is an ideal.

(g) (2 points) Show that $\mathsf{R}/\mathfrak{n}(\mathsf{R})$ does not have any nilpotent elements.

(h) (2 points) Show that the intersection of all prime ideals of R, that is $\displaystyle\bigcap_{\wp \leq \mathsf{R},\, \wp \text{ prime}} \wp$, contains $\mathfrak{n}(\mathsf{R})$. (`Hint:` Show that every prime ideal contains $\mathfrak{n}(\mathsf{R})$.)

Question 2 (8 points)

Let $R = \mathbf{Z}[\sqrt{5}]$. For $\alpha = a + b\sqrt{5} \in R$ define the *norm* function $N(\alpha) = a^2 - 5b^2$.

(a) (2 points) Assuming that N is a multiplicative function, i.e. $N(\alpha \cdot \beta) = N(\alpha)N(\beta)$, classify the units of R.

(b) (2 points) Deduce that the elements 2, $3 + \sqrt{5}$ and $3 - \sqrt{5}$ (of R) are irreducible in R. (`Hint:` What are squares $\pmod 5$?)

(c) (2 points) Deduce that R is not a UFD.

(d) (2 points) Determine the ring $\mathsf{S} = \mathbf{Z}[\sqrt{5}]/\langle 2 \rangle$, i.e find a ring which is isomorphic to S.
(`Hint:` What do you expect from S to be? Determine all the equivalence classes and guess.)