Université Galatasaray, Département de Mathématiques
2018 - Spring Semester – Math 513 - Algebraic Geometry
Final Exam, 27 May 2019 – Ayberk Zeytin, 120 minutes
Name & Surname: ________________________________ **Sign:** ______________

Question:	1	2	3	Total
Points:	2	10	6	18
Score:				

Question 1 (2 points)
Use the fact that the class number of the number field $\mathbf{Q}(\sqrt{5})$ is 1 to show that the solutions of the equation

$$t^2 - 5u^2 = 4$$

are given by $y = \pm F_{2n}$ and $t = \pm(F_{2n-1} + F_{2n+1})$; where F_n is the n^{th} Fibonacci number : $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Question 2 (10 points)

Let $\mathfrak{p} = (4, \frac{3+\sqrt{-71}}{2})$ and $\mathfrak{q} = (3, \frac{1+\sqrt{-71}}{2})$.

(a) (2 points) Show that $\mathfrak{p}\mathfrak{q} = (12, \frac{-5+\sqrt{-71}}{2})$.

(b) (2 points) Interpret the result in terms of corresponding binary quadratic forms. Show that the reduced form corresponding to $\mathfrak{p}\mathfrak{q}$ is $(2, 1, 9)$.

(c) (2 points) Show that the form $(2, 1, 9)$ represents 120. (`Hint:` This is a positive definite form: there are only finitely many cases to consider. Write your trials, as well.)

(d) (2 points) Assuming that the ring of integers of a number field is a Dedekind domain, hence ideal factorisation is unique, show $\mathbf{Q}(\sqrt{6})$ is not a UFD. (`Hint:` For the ideals $\mathfrak{p} = (2, 4+\sqrt{6})$ and $\mathfrak{q} = (5, 4+\sqrt{6})$, compute $\mathfrak{p}^2$, $\mathfrak{q}\bar{\mathfrak{q}}$, $\mathfrak{p}\mathfrak{q}$ and $\mathfrak{p}\bar{\mathfrak{q}}$.)

(e) (2 points) Interpret the result in terms of corresponding binary quadratic forms.

Question 3 (6 points)

(a) (2 points) Show that for any integers x_1, x_2, y_1, y_2 and D, we have :

$$(x_1^2 + Dy_1^2)(x_2^2 + Dy_2^2) = (x_1x_2 - Dy_1y_2)^2 + D(x_1y_2 + x_2y_1)^2$$

(b) (2 points) Interpret this identity in terms of composition defined on the group of binary quadratic forms of appropriate discriminant.

(c) (2 points) This identity is very old: in fact due to Bharmagupta (600s). Explain what Gauss' composition means by building on this identity. (`Hint:` Refer to Question 2 : the form $f = (4, 3, 5)$ represents 12, the form $g = (3, 1, 6)$ represents 10, hence $fg = (12, -5, 2)$ should represent 120.)