Université Galatasaray, Département de Mathématiques
2020 - Fall Semester – Math 519 - Complex Geometry
Final Exam, 06 January 2020 – Ayberk Zeytin, 120 minutes
Name & Surname: ______________________ **Sign:** __________

Question:	1	2	Total
Points:	12	14	26
Score:			

Question 1 (12 points)

We fix an odd prime number p and let $\zeta_p = e^{2\pi\sqrt{-1}/p}$. The aim of this question is to factorize the ideal (p) in the cyclotomic extension $\mathbf{Q}(\zeta_p)/\mathbf{Q}$. We admit (without proof) the fact that the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\zeta_p)$ is $\mathbf{Z}[\zeta_p]$. You may also need the fact that the extension $\mathbf{Q}(\zeta_p)/\mathbf{Q}$ is a Galois extension. The Galois group's i^{th} element is the homomorphism determined by sending ζ_p to ζ_p^i for $i = 1, 2, \ldots, p-1$.

(a) (2 points) Prove Eisenstein's irreducibility criterion : let $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \in \mathbf{Z}[X]$ be a monic polynomial so that $p \mid a_i$ for each $i = 0, 1, \ldots, n-1$ but $p^2 \nmid a_0$. Show that $f(X)$ is irreducible. (`Hint:` Prove by contradiction. You'll need Gauß Lemma.)

(b) (2 points) Show that for each $i = 1, 2, \ldots, p-1$ we have $p \mid \binom{p}{i} = \frac{p!}{i!(p-i)!}$. ($\texttt{Hint:}$ Use induction.)

(c) (2 points) Show that the polynomial $f(X) = 1+X+\ldots+X^{p-1}$ is the minimal polynomial of ζ_p and deduce that the degree of the extension $\mathbf{Q}(\zeta_p)/\mathbf{Q}$. ($\texttt{Hint:}$ Show that $f(X+1)$ is irreducible using (a) and (b).)

(d) (2 points) Show that for and $g = 1, 2, \ldots, p-1$ the element $\varepsilon_g := \frac{1-\zeta_p^g}{1-\zeta_p}$ is a unit of $\mathbf{Z}[\zeta_p]$. (`Hint:` Show that both ε_g and ε_g^{-1} are integral over $\mathbf{Z}$.)

(e) (2 points) Show that p and $(1-\zeta_p)^{p-1}$ are associates, i.e. one is a unit times the other. (`Hint:` Evaluate the minimal polynomial of ζ_p at $X = 1$ to get $p = \prod_{i=1}^{p-1}(1-\zeta_p^i)$. Try to express the latter product in terms of ε_g)

(f) (2 points) Show that $\mathbf{Z}[\zeta_p]/(1-\zeta_p) \cong \mathbf{Z}/p\mathbf{Z}$. (`Hint:` Do not try to find an explicit homomorphism. Instead use the fundamental formula relating the extension degree to ramification index and residual degree.)

Question 2 (14 points)

We consider the polynomial $f(X, Y) = Y^2 - X^3 - X^2 \in \mathbf{C}[X, Y]$, set $C_f = \mathbf{C}[X, Y]/(f)$, $\mathbf{C}(f) := ff(C_f)$ and Z_f be the zero set of f in $\mathbf{C}^2$.

(a) (2 points) Show that f is irreducible. (Hint: You may try to solve this directly or prove a generalized Eisenstein criterion.)

(b) (2 points) Show that the ring C_f is not integrally closed.(Hint: Consider the element $\frac{Y}{X}$.)

(c) (2 points) Show that the integral closure of C_{f} in $\mathbf{C}(\mathsf{f})$ is $\mathbf{C}[\mathsf{X}][\mathsf{Y}/\mathsf{X}]$.

(d) (2 points) Does there exist an $a \in \mathbf{C}$ so that $(X - a)$ is a unit?

(e) (2 points) Does there exist an $a \in \mathbf{C}$ so that $(X - a)$ is a unit, if we replace $\mathbf{C}$ by $\mathbf{R}$?

(f) (2 points) Show the following inclusions :

$$\mathbf{C}[X] \subset C_f \subset \mathbf{C}[X, Z]/(Z^2 - (X+1))$$

(g) (2 points) Determine the maps determined by the above inclusions and describe the geometrically your maps (i.e. real picture).