MATH 513
EXERCISES 1

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(1) ► Show that the series $\sum_{n=1}^{\infty} \frac{1}{n} e^{-ns}$ converges at 0 but diverges at $\pi\sqrt{-1}$.
► Show that if a Dirichlet series converges and diverges at different points on the same vertical line, then this line must be the line of convergence. Deduce that the line of convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n} e^{-ns}$ is $\sigma_o = 0$.
► Verify this by using the formula given in the lecture.

(2) A Dirichlet series is absolutely convergent if the series

$$\sum_{n=1}^{\infty} |a_n e^{-\lambda_n s}|$$

is convergent.
► Show that an absolutely convergent Dirichlet series is convergent.
► Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (\log(n))^{-s}$ is convergent for all s.
► Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (\log(n))^{-s}$ is not absolutely convergent for any s.
► Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be an ordinary Dirichlet series with σ_c being its abscissa of convergence. Let σ_a denote the abscissa of absolute convergence of $f(s)$. Show hat $\sigma_a \leq \sigma_c + 1$.

(3) Show, mimicing the technique in class, that

$$\sum_{n=1}^{N-1} \left| e^{\lambda_n s} - e^{\lambda_{n+1} s} \right| \leq \frac{|s|}{\sigma} e^{\sigma \lambda_N}.$$

(4) Let $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ be a Dirichlet series. Assume that there is some $s_o \in \mathbf{C}$ with $\sigma_o > 0$ for which the given series converges, and the series diverges for all s with $\sigma < 0$. Then

$$\sigma_a = \limsup_{N \to \infty} \frac{\log(\sum_{n=1}^{N} |a_n|)}{\lambda_N}$$

is the absolute abscissa of convergence.

(5) Show that if $\sigma_c(f)$ (resp. $\sigma_c(g)$) is the abscissa of convergence of $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ (resp. $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$) then the series $f(s)g(s)$ converges at least where :
i. both series converges, and
ii. at least one converges absolutely.

(6) For a positive integer k, let $\sigma_k(n) := \sum_{d|n} d^k$, i.e. the sum of k^{th} powers of positive divisors of n. Show that :

$$\sum_{n=1}^{\infty} \sigma_k(n) n^{-s} = \zeta(s)\zeta(s-k) \quad \text{for and } s \text{ with } \sigma > k+1.$$