MATH 513
EXERCISES 2

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(1) Define $f(s) = \Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$.
- ▶ Show that $f(s+1) = \frac{s}{2} f(s)$.
- ▶ Deduce that $2^{s+1} f(s+1) = s2^2$ and observe that $2^s f(s)$ satisfies the property that we expect $\Gamma(s)$ to satisfy, too!
- ▶ Show that $2^s f(s) = C\Gamma(s)$; where $C = \lim_{N\to\infty} 2^N \sqrt{N} \frac{(N-1)!^2}{(2N-1)!}$. (irrelevant exercise : $C = 2\sqrt{\pi}$).

(2) In this exercise, we'll find the integral representation of $\Gamma(s)$, that is, we'll prove $\Gamma(s) = h(s) := \displaystyle\int_0^\infty t^{s-1} e^{-t}\, dt$, for all $\sigma > 0$.
- ▶ Show that the integral on the right hand side (i.e. $h(s)$) converges for $\sigma > 0$.
- ▶ Show that $h(s+1) = sh(s)$ by using integration by parts.
- ▶ Show that $h(1) = 1$.
- ▶ Verify by integration by parts the equality :
$$\int_0^N \left(1 - \frac{t}{N}\right)^N t^{s-1}\, dt = N^s \sum_{r=0}^{N} \frac{(-1)^r \binom{N}{r}}{r+s}$$
- ▶ Show by comparing the poles of two complex functions of s that
$$N^s \sum_{r=0}^{N} \frac{(-1)^r \binom{N}{r}}{r+s} = \frac{N}{N+s} \Gamma_N(s);$$
where $\Gamma_N(s) = \frac{N^s (N-1)!}{s(s+1)\ldots(s+N-1)}$.
- ▶ Deduce that $h(s) = \Gamma(s)$ by calculating limit and $N \to \infty$.

(3) Determine all the characters of the groups $\mathbf{Z}/8\mathbf{Z}$, $\mathbf{Z}/12\mathbf{Z}$, $\mathbf{Z}/15\mathbf{Z}$. Write the explicit isomorphisms between groups and the groups of characters.

(4) Determine the group $\mathbf{Z}/p^2\mathbf{Z}$; where p is a prime number and determine all its characters.

(5) Let ϕ denote the Euler's totient function.
- ▶ Show that ϕ is multiplicative, that is $\phi(nm) = \phi(n)\phi(m)$ when $(m, n) = 1$.
- ▶ Recall that a function f is called completely multiplicative if $f(nm) = f(n)f(m)$ for any $n, m \in \mathbf{Z}$. Give an example to show that ϕ is not completely multiplicative.
- ▶ Let p be a prime number. Show that $\phi(p^k) = p^k \left(1 - \dfrac{1}{p}\right)$.
- ▶ Deduce that $\phi(N) = N \displaystyle\prod_{p|N} \left(1 - \frac{1}{p}\right)$.

(6) Show that the Legendre symbol is indeed a Dirichlet character.

(7) Let χ be a Dirichlet character and let $a, b \in \mathbf{Z}$ be arbitrary. Show that if $na \equiv b \pmod N$, then $\chi(n) = \chi(b)\overline{\chi}(a)$; where $\overline{\chi}$ denotes the conjugate character.

(8) Let $\overline{n} \in (\mathbf{Z}/N\mathbf{Z})^\times \setminus \{\overline{1}\}$ be an arbitrary element. Show that there is always a Dirichlet character, say χ, modulo N so that $\chi(\overline{n}) \neq 1$.

(9) Let χ be a real valued Dirichlet character modulo q. Define

$$f(n) = \sum_{d|n} \chi(d).$$

- Show that $f(1) = 1$ and $f(n) \geq 0$ for all $n \in \mathbf{N}$
- Show that $f(n) \geq 1$ when n is a perfect square.

(10) By using induction on the integer r show that :

- $(\mathbf{Z}/p^r\mathbf{Z})^\times$ is cyclic for any prime number $p > 2$
- if $n \equiv 1 \pmod 8$ then $n \equiv x^2 \pmod{2^r}$ for $r \geq 3$.

Show using the previous results that if

- $(\mathbf{Z}/p^r\mathbf{Z})^\times \cong \mathbf{Z}/(p^{r-1}(p-1))\mathbf{Z}$ if p is an odd prime.
- $(\mathbf{Z}/p^r\mathbf{Z})^\times \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2^{r-2}\mathbf{Z}$.