MATH 513
EXERCISES 4

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(1) Let $Q\colon \mathbf{R}^r \to \mathbf{R}$ be any quadratic form. Show that the function
$$B(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$$
is a bilinear form.

(2) Calculate the bilinear form $B(x,y)$ associated to the following quadratic forms :
- $Q(x,y) = x^2 + 3xy + 3y^2$
- $Q(x,y,z) = x^2 + y^2 + yz + z^2$

(3) Consider the (more general) theta series :
$$\theta(z,\tau) := \sum_{n\in\mathbf{Z}} e^{i\pi n^2\tau + 2\pi i z n}$$
- Show that if one fixes z, then the series $\theta(z,\tau)$ converges for all $\tau \in \mathbb{H}$.
- Show that if one fixes τ, then the series $\theta(z,\tau)$ converges for all $z \in \mathbf{C}$.
- Deduce that the series is convergent for all $(z,\tau) \in \mathbf{C}\times\mathbb{H}$.
- Show that $\theta(z+1,\tau) = \theta(z,\tau)$, and therefore one may talk about its Fourier expansion in the variable z.
- Show that $\theta(z+\tau,\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z,\tau)$. The factor $e^{-\pi i\tau-2\pi i z}$ is called a theta factor of this theta series.

(4) Let $Q(x,y) = x^2 + y^2$. Verify the following equality :
$$\sum_{x,y\in\mathbf{Z}} e^{2\pi i(x^2+y^2)\tau} = \left(\sum_{n\in\mathbf{Z}} e^{2\pi i n^2\tau}\right)^2$$

(5) Show that the Mellin transform of the theta series :
$$\sum_{n\in\mathbf{Z}} e^{-\pi n^2 t}, \quad (t\in\mathbf{R})$$
is the integral representation of the completed ζ function. Hint: Integrate termwise.

(6) Explain why we assumed a Dirichlet character to be primitive when defining the corresponding Gauss sum, by investigating what happens when the Dirichlet character is not primitive.

(7) Given a primitive Dirichlet character modulo q, we define
$$\tau(n,\chi) = \sum_{a\in\mathbf{Z}/q\mathbf{Z}} \chi(a)e^{2\pi i a n/q}$$
Show that if $(n,q) = 1$ then $\tau(n,\chi) = \overline{\chi}(n)\tau(\chi)$.

(8) Let χ_1, χ_2 be two primitive Dirichlet characters modulo q_1 and q_2, respectively. Set $\chi = \chi_1\chi_2$. Show that :
$$\tau(\chi) = \tau(\chi_1)\tau(\chi_2)\chi_1(q_2)\chi_2(q_1);$$
where τ denotes the corresponding Gauss sum.