MATH 513
EXERCISES 5

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(1) For an algebraic number $\alpha \in \mathbf{C}$ show that the set $I_\alpha = \{p(t) \in \mathbf{Q}[t] \mid p(\alpha) = 0\}$ is an ideal of $\mathbf{Q}[t]$.

(2) Explain why $t^n - 1$ is not the minimal polynomial of $\zeta_n = e^{2\pi\sqrt{-1}/n}$ when n is not a prime number.

(3) Determine all a, b, c and d so that the complex number $\frac{a+b\sqrt{d}}{c}$ is an algebraic integer.

(4) In this exercise, we will compute the discriminant of a quadratic number field.
- ▶ Determine $\mathbf{Z}_K$ when $K = \mathbf{Q}(\sqrt{d})$ for some square-free integer d.
- ▶ Determine an integral basis of $\mathbf{Z}_K$.
- ▶ Use the basis you obtained and calculate the discriminant of $\mathbf{Q}(\sqrt{d})$.

(5) Find a square-free integer d so that $\mathbf{Z}_K$ is not a UFD, and prove your claim.

(6) Fix a number field K.
- ▶ Show that the product of two fractional ideals is again a fractional ideal.
- ▶ Show that if $\mathfrak{A}$ is a fractional ideal of $\mathbf{Z}_K$, then
$$\mathfrak{A}' = (\mathbf{Z}_K : \mathfrak{A}) := \{\alpha \in K \mid \alpha\mathfrak{A} \subseteq \mathbf{Z}_K\}$$
is also fractional ideal of K.
- ▶ Determine the product $\mathfrak{A} \cdot \mathfrak{A}'$.

(7) This exercise demonstrates the fact that being a Dedekind domain is of crucial importance for our considerations. Let $R = \mathbf{Z}[\sqrt{-3}]$.
- ▶ Show that the ideal generated by 2 in R cannot be written as a product of prime ideals. Deduce that R is not a Dedekin domain.
- ▶ Set $\mathfrak{A} = (2, 1 + \sqrt{-3}) \subseteq R$. Show that $\mathfrak{A}$ is not principal and determine $\mathfrak{A}'$. Deduce that R is not a Dedekin domain.

(8) Let a, b, c and d are integers with d being square-free. Let $K = \mathbf{Q}(\sqrt{d})$. Set $\mathfrak{A} = (a, b + c\sqrt{d})$.
- ▶ Show that the class of $\mathfrak{A}' \in H(K)$ contains $(a, b - c\sqrt{d})$.
- ▶ Deduce that if $\mathfrak{A}$ is any fractional ideal of K, then the class of $\mathfrak{A}' \in H(K)$ contains $\overline{\mathfrak{A}}$; where $\bar{\cdot}$ is the unique non-trivial element of the Galois group $K/\mathbf{Q}$.

(9) Notice that if I is an ideal of $\mathbf{Z}_K$ of norm 1, then $I = \mathbf{Z}_K$. Deduce that if $\mathfrak{A}$ is an ideal of $\mathbf{Z}_K$ whose norm is a prime number, then $\mathfrak{A}$ is a prime ideal.