MATH 513
EXERCISES 6

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(1) Prove the Euler product formula for the Hecke L-series of a number field K corresponding to the ideal class character χ; namely :

$$L_K(\chi, s) := \sum_{I} \frac{\chi(I)}{N(I)} = \prod_{\mathfrak{p}:\text{prime ideal}} \frac{1}{1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s}}$$

(2) Show that if $\{\omega_1, \omega_2\}$ and $\{\eta_1, \eta_2\}$ are two bases of $K/\mathbf{Q}$, then $\{\omega_1\eta_1, \omega_1\eta_2\}$ is again a basis of $K/\mathbf{Q}$.

(3) Let R be an integral domain, I be an ideal of R. Set $K = \mathrm{ff}(R)$. Prove of disprove the following statement : For any $\omega \in R$ there is an element α so that $\alpha\omega \in I$.

(4) Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be lattices in a quadratic number field K. Show that :
- $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c})$
- $(\mathfrak{a} : (\mathfrak{b} + \mathfrak{c})) = (\mathfrak{a} : \mathfrak{b}) \cap (\mathfrak{a} : \mathfrak{c})$

(5) Let K be a quadratic number field. For any two elements $\xi_1, \xi_2 \in K$, we define $\partial(\xi_1, \xi_2) := (\xi_1\iota(\xi_1) - \iota(\xi_1)\xi_2)^2$; where ι denotes the unique non-trivial automorphism of K.
- Prove that $\partial(\xi_1\xi_2) = 0$ if and only if ξ_1 and ξ_2 are $\mathbf{Q}$-linearly dependent.
- Set $\mathfrak{a} = [\omega_1, \omega_2]$ be a lattice in K. Prove that if $\mathfrak{a} = [\eta_1, \eta_2]$ is another basis for the lattice $\mathfrak{a}$, the $\partial(\omega_1, \omega_2) = \partial(\eta_1, \eta_2)$. Deduce that one can define the discriminant of a lattice $\mathfrak{a}$ in K as $\partial(\xi_1, \xi_2)$; where $\{\xi_1, \xi_2\}$ is any generating set of $\mathfrak{a}$.
- If $\mathfrak{a}$ is any lattice in K, then for any $\lambda \in K$, we have $\partial(\lambda\mathfrak{a}) = \lambda^2\partial(\mathfrak{a})$.
- Let $\mathfrak{a}, \mathfrak{b}$ be two lattices in K. If $\mathfrak{a} \subseteq \mathfrak{b}$ then :

$$\partial(\mathfrak{a}) = \partial(\mathfrak{b})[\mathfrak{b} : \mathfrak{a}]^2$$

- Let Δ be a quadratic discriminant. Show that $\partial(\mathcal{O}_\Delta) = \partial([1, \omega_\Delta]) = \Delta$.