MATH 513
EXERCISES 7

A. ZEYTİN

(1) Let Δ be a quadratic discriminant, $d, e \in \mathbf{N}$ and set $f = de$. Prove that
- $\mathfrak{a} = d\mathcal{O}_\Delta \cap \mathcal{O}_{\Delta f^2}$ is an ideal of $\mathcal{O}_{\Delta f^2}$
- $\mathcal{R}(\mathfrak{a}) = \mathcal{O}_{\Delta e^2}$

(2) Let Δ be a quadratic discriminant and let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional $\mathcal{O}_\Delta$-ideals. Show that the followings are fractional $\mathcal{O}_\Delta$-ideals, too :
- $\lambda\mathfrak{a}$; where $\lambda \in K \setminus \{0\}$,
- $\mathfrak{a} \cap \mathfrak{b}$,
- $\mathfrak{a} + \mathfrak{b}$.

(3) Let Δ be a quadratic discriminant and let $\mathfrak{a}$ and $\mathfrak{b}$ be two invertible fractional $\mathcal{O}_\Delta$-ideals. Show that :
- $\mathfrak{a}\,\mathfrak{b}$ is also an invertible fractional $\mathcal{O}_\Delta$-ideal.
- $(\mathcal{O}_\Delta : \mathfrak{a}\mathfrak{b}) = (\mathcal{O}_\Delta : \mathfrak{a})(\mathcal{O}_\Delta : \mathfrak{b})$

(4) Calculate the product

$$\left[6, \frac{1+\sqrt{97}}{2}\right]\left[18, \frac{5+\sqrt{9}}{2}\right]$$

(5) Let Δ be a quadratic discriminant and say :

$$\mathfrak{a}_1 = \left[a_1, \frac{b_1+\sqrt{\Delta}}{2}\right] \quad \text{and} \quad \mathfrak{a}_2 = \left[a_2, \frac{b_2+\sqrt{\Delta}}{2}\right];$$

where $a_1, a_2 \in \mathbf{N}$, $b_1, b_2 \in \mathbf{Z}$ and $\gcd(a_1, a_2, \frac{b_1+b_2}{2}) = 1$. Let b be any integer satisfying $b \equiv b_1 \pmod{2a_1}$ and $b \equiv b_2 \pmod{2a_2}$. Show that

$$\mathfrak{a}_1\mathfrak{a}_2 = \left[a_1 a_2, \frac{b+\sqrt{\Delta}}{2}\right]$$

Verify your answer for Exercise 4 using this formula.

(6) Let Δ be a quadratic discriminant and $\mathfrak{a}$ a fractional $\mathcal{O}_\Delta$-ideal. Prove that the followings statements are equivalent :
- $\mathfrak{a}$ is invertible
- for every fractional $\mathcal{O}_\Delta$-ideal $\mathfrak{b}$, there exists a (necessarily unique) fractional $\mathcal{O}_\Delta$-ideal $\mathfrak{c}$ so that $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$
- for every fractional $\mathcal{O}_\Delta$-ideal $\mathfrak{b}$ $(\mathcal{O}_\Delta : \mathfrak{a})\mathfrak{b} = (\mathfrak{b} : \mathfrak{a})$
- for any given fractional $\mathcal{O}_\Delta$-ideals $\mathfrak{b}$ and $\mathfrak{c}$, if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{c}$ then $\mathfrak{b} \subseteq \mathfrak{c}$.

(7) Let Δ be a quadratic discriminant and let $\mathfrak{a}$ be a fractional $\mathcal{O}_\Delta$-ideal. Prove the followings "almost unicity" theorem concerning the generators : If there are $a, \widetilde{a}, e, \widetilde{e} \in \mathbf{N}$ and $b, \widetilde{b} \in \mathbf{Z}$ so that

$$\mathfrak{a} = e\left[a, \frac{b+\sqrt{\Delta}}{2}\right] = \widetilde{e}\left[\widetilde{a}, \frac{\widetilde{b}+\sqrt{\Delta}}{2}\right]$$

then we necessarily have $a = \widetilde{a}$, $e = \widetilde{e}$ and $b \equiv \widetilde{b} \pmod{2a}$

(8) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two lattices in K and R be an order in K so that $R = \mathcal{R}(\mathfrak{a}) \cap \mathcal{R}(\mathfrak{b})$. A map $\varphi\colon \mathfrak{a} \to \mathfrak{b}$ is called an R-isomorphism if φ is a group isomorphism satisfying $\varphi(\lambda\alpha) = \lambda\varphi(\alpha)$ for all $\alpha \in \mathfrak{a}$ and $\lambda \in R$. Show that the lattice define the same class in the class group $\mathcal{C}_\Delta$ if and only if there exists an R-isomorphism $\varphi\colon \mathfrak{a} \to \mathfrak{b}$.