MATH 513
EXERCISES 8

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In this set of exercises, Δ stands for a quadratic discriminant, and $f = (a, b, c) = ax^2 + bxy + cy^2$ stands for a binary quadratic form of discriminant Δ. F_Δ denotes the set of all primitive binary quadratic forms of discriminant Δ.

(1) Show that if Δ is square-free then f must be primitive but not conversely, that is there are primitive quaratic forms with non-squarefree discriminant.

(2) Show that f is
- degenerate if and noly if f assumes both positive and negative values and $f(X, Y) = 0$ for infinitely many $(X.Y) \in \mathbf{Z}^2$.
- positive (resp. negative) definite if and only if f assumes only non-negative (resp. non-positive) values and further $f(X, Y) = 0$ only for $(X, Y) = (0, 0)$.
- indefinite if and only if f assumes both positive and negative values and $f(X, Y) = 0$ only for $(X, Y) = (0, 0)$.

(3) Show that the map

$$\begin{aligned} \mathrm{GL}(2, \mathbf{Z}) \times F_\Delta &\to F_\Delta \\ (A, f) &\mapsto A \cdot f \end{aligned}$$

is indeed a group action. Show further that the equivalence relation induced by this action, namely $f \sim f'$ if and only if $f' = A \cdot f$ for some $a \in \mathrm{GL}(2, \mathbf{Z})$ is an equivalence relation.

(4) In this exercise, we will show that $\mathrm{PGL}(2, \mathbf{Z})$ is generated by

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} L = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\mathrm{PSL}(2, \mathbf{Z})$ is generated by the latter two (i.e. S and L).
- Determine the orders of U, S and L in $\mathrm{PGL}(2, \mathbf{Z})$.
- Using induction show that for any $n \in \mathbf{Z}$, $(LS)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
- Given any $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{PGL}(2, \mathbf{Z})$, express SA and $(LS)^n A$ in terms of p, q, r and s.
- Write $\begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \in \mathrm{PSL}(2, \mathbf{Z})$ as a product of S and L.
- Assume now that $r \neq 0$. Investigating the cases $|p| \geq |r|$ and $|p| < |r|$ cases separately and using the division algorithm show that there is a sequence of integers $n_1, n_2, \ldots, n_k$ with the property that either
$$A = S\,(LS)^{n_1}\,S\,(LS)^{n_2} \cdots S\,(LS)^{n_k} \quad \text{or} \quad A = (LS)^{n_1}\,S\,(LS)^{n_2} \cdots S\,(LS)^{n_k}$$
- Conclude that $\mathrm{PSL}(2, \mathbf{Z})$ is generated by S and L freely.
- Deduce that $\mathrm{PGL}(2, \mathbf{Z})$ is generated by U, S and L. Show that this set does not generate $\mathrm{PGL}(2, \mathbf{Z})$ freely.

(5) Let $f = (195751, 1212121, 187641)$.
- Find the normalization of f and notice that the normalization is a normal form..
- Show that $(1, 1, 1) \in [f]^+$.

(6) Let $\Delta < 0$. If f is a positive (resp. negative) definite form then for any $A \in \mathrm{PSL}(2, \mathbf{Z})$ $A \cdot f$ is again a positive (resp. negative) definite form.

(7) Determine all normal forms for negative quadratic discriminants greater than oe equal to -12. Which binary quadratic forms in your list are reduced?

(8) Apply reduction algorithm to $f = (33824333, 889961, 5854)$.

(9) Find all the reduced integral forms of discriminant 5, 8, 12, 13.