MATH 504
EXERCISES 1

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Unless otherwise stated G is a group.

(1) Study the properties (that is associativity, commutativity, existence of identity and inverses) of the following binary operations :
- $n * m = n + m + nm$ on $\mathbf{Z}$
- $x * y = \sqrt{xy}$ on $\mathbf{R}_+$
- $z * w = |zw|$ on $\mathbf{C}$

(2) Prove or disprove the following statements :
- Every binary operation on a set containing only one element is associative.
- Every binary operation on a set containing only one element is commutative.

(3) Let $G = C^0(\mathbf{R})$. On G we define the usual pointwise addition, pointwise multiplication and composition on G as follows :

$$+ \colon G \times G \to G$$
$$(f, g) \mapsto f + g(x) := f(x) + g(x)$$

$$\cdot \colon G \times G \to G$$
$$(f, g) \mapsto f \cdot g(x) := f(x)g(x)$$

$$\circ \colon G \times G \to G$$
$$(f, g) \mapsto f + g(x) := f(g(x))$$

Prove or disprove the following statements :
- $(G, +)$ is associative
- $(G, -)$ is commutative
- $(G, -)$ is associative
- $(G, \cdot)$ is associative
- $(G, \cdot)$ is commutative
- $(G, \circ)$ is commutative

(4) In a group $(G, \cdot)$ show that left and right cancellation holds, i.e.
- (left cancellation) if $x \cdot h = g \cdot h$ then $x = g$
- (right cancellation) if $h \cdot x = h \cdot g$ then $x = g$

(5) Let $(G, \cdot)$ be a group. Show that for any $g, h \in G$ the following equations have unique solutions :
- $x \cdot g = h$
- $g \cdot x = h$

(6) Let $(G, *)$ be a group. Show that G has a unique idempotent element, that is and element $x \in G$ so that $x^2 = x$.

(7) Show that if G is a group satisfying $g * g = e$ for all $g \in G$, then G is abelian.

(8) Show that if G is a group satisfying $g * g * g = e$ for all $g \in G$, then G is abelian.

(9) Decide whether the following subsets are subgroups of $G = \mathrm{GL}(n, \mathbf{R})$:
- $H = \{M \in G \mid \det(M) = 2\}$
- $H = \{M \in G \mid \det(M) = \pm 1\}$
- $H = \{M \in G \mid M \text{ is diagonal}\}$
- $H = \{M \in G \mid M \text{ has no zeros on the diagonal}\}$

(10) Consider the set $G = C^0(\mathbf{R})$ as a group under pointwise multiplication :

- $H = \{f \in G \mid f(0) = 0\}$
- $H = \{f \in G \mid f(0) = 1\}$
- $H = \{f \in G \mid f(1) = 1\}$
- $H = \{f \in G \mid f \text{ is constant}\}$

(11) Let G be a group and H and K are subgroups. Show that

- $H \cap K$ is a subgroup, and
- $H \cup K$ is not necessarily a subgroup.