MATH 504
EXERCISES 10

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Unless otherwise stated R is a ring.

(1) Determine if the following maps are ring homomorphisms or not. If yes, determine $\ker(\varphi)$ and $\mathrm{im}(\varphi)$.
- ▶ for any non-zero integer n :
$$\begin{aligned}\varphi_n \colon \mathbf{Z} &\to \mathbf{Z}\\ a &\mapsto n \cdot a\end{aligned}$$
- ▶ for any positive integer n :
$$\begin{aligned}\varphi_n \colon M_n(\mathbf{R}) &\to \mathbf{R}\\ M &\mapsto \det(M)\end{aligned}$$
- ▶ for any positive integer n :
$$\begin{aligned}\varphi_n \colon M_n(\mathbf{R}) &\to \mathbf{R}\\ M &\mapsto \mathrm{tr}(M)\end{aligned}$$
- ▶
$$\begin{aligned}\varphi \colon M_2(\mathbf{R}) &\to \mathbf{R}\\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto r + s\end{aligned}$$
- ▶
$$\begin{aligned}\varphi_n \colon M_2(\mathbf{R}) &\to \mathbf{R}\\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto p\end{aligned}$$

(2) Let $\varphi : R \to S$ be a ring homomorphism. Decide whether the following statements hald true. If yes, give a proof, else give a counter-example :
- ▶ If $u \in R$ is a unit of R (i.e. an invertible element), then $\varphi(u)$ is a unit of S.
- ▶ If $\varphi(u)$ is a unit of S then $u \in R$ is a unit of R.
- ▶ If $v \in R$ is a unit of S, then $\varphi^{-1}(v)$ is a unit of R.
- ▶ If $r \in R$ is a zero divisor then $\varphi(r)$ is a zero divisor of S. Is the converse true?
- ▶ If $\varphi(u)$ is a zero divisor of S then $u \in R$ is a zero divisor of R.
- ▶ If $s \in S$ is a zero divisor then each element of the set $\varphi^{-1}(s)$ is a zero divisor of R.

(3) Prove that $\mathbf{R}[X]/(X^2+1)$ and $\mathbf{C}$ are isomorphic. Deduce that the ideal (X^2+1) is a maximal ideal.

(4) Consider the polynomial $p(X) = X^2 + 1 \in (\mathbf{Z}/3\mathbf{Z})[X]$.
- ▶ Explicitly write each element of the ring $R = (\mathbf{Z}/3\mathbf{Z})[X]/(p(X))$.
- ▶ Show that R is an integral domain.
- ▶ Find the multiplicative inverse of $X + (p(X)) \in R$.
- ▶ By finding multiplicative inverses of remaining non-zero elements of R, show that R is a field. Deduce that $(p(X))$ is a maximal ideal.
- ▶ Are R and $\mathbf{Z}/9\mathbf{Z}$ isomorphic?

(5) Prove that the rings $\mathbf{R}[X]$ and $\mathbf{Z}[X]$ are not isomorphic.

(6) Let R be a ring and I and J are ideal of R so that $I \subseteq J$.

- ► Show that the map

$$\begin{aligned} \varphi : R/I &\longrightarrow R/J \\ r + I &\longmapsto r + J \end{aligned}$$

  is a well-defined surjective ring homomorphism.
- ► Show that $\ker(\varphi) = I/J$

(7) Show that the rings $\mathbf{Q}[\sqrt{-5}]$ and $\mathbf{Q}[X]/(X^2-2x+6)$ are isomorphic. Are the rings $\mathbf{Z}[\sqrt{-5}]$ and $\mathbf{Z}[X]/(X^2-2x+6)$ isomorphic?

(8) Let R, S be two rings, $\varphi : R \longrightarrow S$ be a ring homomorphism.
- ► If $\mathfrak{p}$ is a prime ideal of R and φ is surjective then $\varphi(\mathfrak{p})$ is a prime ideal of S.
- ► Show that surjectivity of φ is necessary in the previous exercise.
- ► Show that if $\mathfrak{p}$ is a prime ideal of S, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of R.
- ► Show that if $\mathfrak{m}$ is a maximal ideal of R and φ is surjective, then either $\varphi(\mathfrak{m})S$ or $\varphi(\mathfrak{m})$ is a maximal ideal of S.
- ► Show that surjectivity of φ is necessary in the previous exercise.
- ► Show, by an example that $\varphi^{-1}(\mathfrak{m})$ is not necessarily a maximal ideal of R even if $\mathfrak{m}$ is a maximal ideal of S.