MATH 504
EXERCISES 11

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Unless otherwise stated R is a commutative ring with 1.

(1) Show that being associates is an equivalence relation. Determine the equivalence classes of 0 and 1.

(2) Assume that two elements $a, b \in R$ are associates. Show that
- a and b generate the same ideal, that is : $(a) = (b)$.
- a is irreducible if and only if b is irreducible.
- a is invertible if and only if b is invertible.
- a is prime if and only if b is prime.

(3) Let $R_n = (\mathbf{Z}/n\mathbf{Z})[X]$. For the following values of n, show that X is not irreducible.
- 15
- 21
- 22
- 30
- 210

Hint: Try to modify the method used in class for the $n = 6$ case.

(4) Let R be a PID and let I be a non-zero ideal. Show that I is prime if and only if I is maximal.

(5) Let R be an integral domain. Show that R is a UFD if and only if every non-zero prime ideal contains a non-zero principal prime ideal.

(6) Show that the ring $\mathbf{R}[X^2, X^3]$ is not a UFD.

(7) In $\mathbf{Z}[\sqrt{-1}]$ find the factorization of 2 and 13 into irreducibles.

(8) In $\mathbf{Z}[\sqrt{-5}]$ show that the ideal $(3, 2 + \sqrt{-5})$ is not principal.

(9) If R is an integral domain, show that the ideal $I = (X, Y)$ in the ring $R[X, Y]$ is not principal. Show by an example that non-existence of zero divisors is necessary for the claim to hold.

(10) Show that $\mathbf{Z}[X]$ is a UFD but not a PID.

(11) In this exercise, we will prove that if R is a PID, then it is a UFD. R is called *Noetherian* if any given increasing sequence of ideals $I_1 \subseteq I_2, \subseteq I_3 \subseteq \ldots$ in R stabilizes, that is, there is a positive integer n so that $I_n = I_{n+1} = I_{n+2} = \ldots$. Let R be a PID.
- Show that a PID is Noetherian.
- From now on we assume that R is a PID. Show that any non-invertible element of R is divisible by an irreducible element. Hint: Proceed by contradiction.
- Deduce that any $a \in R$ is a product of irreducibles and possibly an invertible element; that is, there are (not necessarily distinct) irreducible elements $p_1, p_2, \ldots, p_n \in R$ and $u \in R^\times$ so that $a = up_1 \cdot p_2 \cdot \ldots \cdot p_n$.
- Show that the above representation is unique up to associates (and reordering).