MATH 504
EXERCISES 12

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Unless otherwise stated R is a commutative ring with 1.

(1) Let R be a UFD and a, b and $a_0, a_1, \ldots, a_n$ be non-zero elements of R.
- ▶ Define a greatest common divisor for a and b. Notice that it is well-defined only up to multiplication by units.
- ▶ Define a greatest common divisor for more than 2 elements, say $a_0, a_1, \ldots, a_n$. Denote it by $\gcd(a_0, \ldots, a_n)$. Notice that it is well-defined only up to multiplication by units. Give an explicit example to see that $\gcd(a_0, \ldots, a_n)$ may not be unique when R fails to be a UFD.
- ▶ Define a least common multiple, $\mathrm{lcm}(a, b)$ for a and b. Notice that it is well-defined only up to multiplication by units.
- ▶ Define a least common multiple for more than 2 elements, say $a_0, a_1, \ldots, a_n$. Denote it by $\mathrm{lcm}(a_0, \ldots, a_n)$. Notice that it is well-defined only up to multiplication by units. Give an explicit example to see that $\mathrm{lcm}(a_0, \ldots, a_n)$ may not be unique when R fails to be a UFD.
- ▶ Verify the identity $ab = \gcd(a, b)\mathrm{lcm}(a, b)$.

(2) Let R be a UFD and $a_0, a_1, \ldots, a_n$ be non-zero elements of R. A non-constant polynomial $f(x) = a_0 + a_1 t + \ldots + a_n x^n \in R[x]$ is called *primitive* if 1 is a greatest common divisor of $a_0, \ldots, a_n$.
- ▶ Decide whether $4x^2 + 7x - 2$ and $4x^3 + 6x - 10$ are primitive in $\mathbf{Z}[x]$.
- ▶ Show that for any non-constant polynomial $f(x) \in R[x]$ there is some element $c \in R$ so that $f(x) = c\, g(x)$ where $g(x) \in R[x]$ is primitive. Show that c is uniquely determined up to multiplication by units. Such a c is called the content of f.
- ▶ Find contents of $4x^2 + 7x - 2$ and $4x^3 + 6x - 10 \in \mathbf{Z}[x]$.
- ▶ Show that if $f(x), g(x) \in R[x]$ are primitive then their product is also primitive. Use induction to deduce that the product of finitely many primitive polynomials is again primitive.
- ▶ Show that if $f(x)$ is a non-constant primitive element of $R[x]$ and $g(x)$ is a divisor of $f(x)$, then $g(x)$ is also primitive.

(3) Let R be a UFD. Show that if I and J are two principal ideals, then $I \cap J$ is also a principal ideal. Find its generator.

(4) Prove that the following rings are not UFDs by finding inequivalent factorizations of appropriate elements :
- ▶ $\mathbf{Z}[\sqrt{-13}]$
- ▶ $\mathbf{Z}[\sqrt{-10}]$
- ▶ $\mathbf{Z}[\sqrt{-6}]$
- ▶ $\mathbf{Z}[\sqrt{10}]$

(5) Factor $14 + 6\sqrt{-1}$ into irreducibles in $\mathbf{Z}[\sqrt{-1}]$.

(6) Show that the ring $R = \{a + b\sqrt{-1} \mid a, b \in \mathbf{Z},\ 2|a \text{ and } 2|n\}$ is not a UFD.

(7) Let K be a field and consider the ring $R = K[X, Y, Z, W]$ of polynomials with coefficients in K in 4 variables.
- ▶ Show that the polynomial $XY - ZW$ is irreducible. Deduce that the ring $R/(XY - ZW)$ is an integral domain.
- ▶ Show that $R/(XY - ZW)$ is not a UFD.