MATH 504
EXERCISES 2

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Unless otherwise stated G is a group.

(1) Show that the intersection of any collection (finite or infinite, countable or uncountable) of subgroups of a group G is again a subgroup. Deduce that for an arbitrary non-empty subset $X \subset G$ the set of all subgroups containing X, denoted $\langle X \rangle$ is a subgroup of G called the subgroup generated by X. Deduce further that if X is itself a subgroup, then the subgroup generated by X is itself, that is show that $\langle X \rangle = X$ when X is a subgroup.
- ▶ Determine the subgroup generated by $X = \{(1\,2), (1\,3)\}$ in $\mathfrak{S}_3$.
- ▶ Determine the subgroup generated by $X = \{(1\,2), (1\,3)\}$ in $\mathfrak{S}_4$.
- ▶ If $X = \{g\}$, then show that the subgroup $\langle X \rangle = \{g^n \mid n \in \mathbf{Z}\}$. Such subgroups are called *cyclic*. If $G = \langle \{g\} \rangle$ for some $g \in G$ then G is called cyclic.
- ▶ Give an example of a finite cyclic subgroup.
- ▶ Give an example of an infinite cyclic subgroup.
- ▶ Let G be a group and $a, b \in G$ be two elements. Try to list all the elements of the subgroup $\langle \{a, b\} \rangle$ if $a^2 = e = b^3$.
- ▶ Let G be a group and $a, b \in G$ be two elements. Try to list all the elements of the subgroup $\langle \{a, b\} \rangle$ if $a^2 = e = b^3$ and $ab = ba$.

(2) Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?
- ▶
$$\begin{aligned} \varphi\colon \mathbf{R}^\times &\to \mathrm{GL}(2, \mathbf{R}) \\ x &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \end{aligned}$$
- ▶
$$\begin{aligned} \varphi\colon \mathbf{R} &\to \mathrm{GL}(2, \mathbf{R}) \\ x &\mapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \end{aligned}$$
- ▶
$$\begin{aligned} \varphi\colon \mathrm{GL}(2, \mathbf{R}) &\to \mathbf{R} \\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto p + s \end{aligned}$$
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- ▶
$$\begin{aligned} \varphi\colon \mathbf{Z} &\to \mathbf{Z} \\ n &\mapsto 504n \end{aligned}$$

(3) Let $n \in \mathbf{N}$ be an integer with $n > 1$. Show that if G is an abelian group, then the map
$$\begin{aligned} \varphi_n\colon G &\to G \\ g &\mapsto g^n \end{aligned}$$
is a group homomorphism. Show further that φ_n need not be a group homomorphism in general.

(4) Show that if G is an abelian group and $\varphi\colon G \to G'$ is a group homomorphism, then $\mathrm{im}(()\varphi)$ is an abelian subgroup of G'.

(5) Let G be a finite group, that is $|G| = n \in \mathbf{N}$. Show that there is an integer m so that $g^m = e$ for all $g \in G$. Show that one may take $m = n$, that is, show that for all $g \in G$ we have $g^{|G|} = e$. Give an example where one may choose $m < |G|$.

(6) Let G be a group, H be a subgroup and N be a normal subgroup of G. Show that $NH = \{nh \in G \mid n \in N,\ h \in H\}$ is a subgroup. Show, by an example, that this fails when N is not normal.

(7) Show that the intersection of two normal subgroups is again a normal subgroup.

(8) Let $\varphi\colon G \to G'$ be a group homomorphism. Show that φ is one-to-one if and only if $\ker(\varphi) = \{e\}$.

(9) Let $\varphi\colon G \to G'$ be a group homomorphism. Define $g_1 \sim g_2$ when $\varphi(g_1) = \varphi(g_2)$.
- ▶ Show that the mentioned relation is an equivalence relation.
- ▶ Describe the equivalence classes of this relation.