MATH 504
EXERCISES 3

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Unless otherwise stated G is a group.

(1) Determine all homomorphisms from :
- ► $\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$
- ► $\mathbf{Z}/7\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$
- ► $\mathbf{Z}/8 \to \mathbf{Z}/7\mathbf{Z}$
- ► $\mathbf{Z}/14\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$
- ► $\mathbf{Z}/7\mathbf{Z} \to \mathbf{Z}/14\mathbf{Z}$
- ► $\mathfrak{S}_3 \to \mathbf{Z}$
- ► $\mathbf{Z} \to \mathfrak{S}_3$

(2) Let m and n be two relatively prime numbers. Show that there is no non-trivial group homomorphism from $\mathbf{Z}/m\mathbf{Z}$ to $\mathbf{Z}/n\mathbf{Z}$.

(3) Determine the automorphism groups of the following groups :
- ► $\mathbf{Z}/4\mathbf{Z}$
- ► $\mathbf{Z}/5\mathbf{Z}$
- ► $\mathbf{Z}/6\mathbf{Z}$

(4) Show that A_n is a normal subgroup of $\mathfrak{S}_n$. (Recall that A_n is the subgroup of even permutations.)

(5) Let G be a group, H and K be subgroups of G.
- ► Show that $H \times K = \{(h, k) \mid h \in H, k \in K\}$ is a group under componentwise multiplication, that is $(h, k) * (h', k') = (hh', kk')$.
- ► Show that the subset $A = \{(h, e) \mid h \in H\}$ is a normal subgroup of $H \times K$.
- ► Show that the subset $B = \{(e, k) \mid k \in K\}$ is a normal subgroup of $H \times K$.
- ► Find a group G' and establish a homomorphism whose kernel is A. Use first isomorphism theorem to deduce that $K \cong (H \times K)/A$
- ► Find a group G'' and establish a homomorphism whose kernel is B. Use first isomorphism theorem to deduce that $H \cong (H \times K)/B$

(6) Let $\varphi\colon G \to G'$ be a group epimorphism and N be a normal subgroup of G. Show that $\varphi(N)$ is a normal subgroup of G'. Show by an example that the claim fails to hold if we do not assume φ to be an epimorphism.

(7) Let $\varphi\colon G \to G'$ be a group homomorphism and N' be a normal subgroup of G'. Show that $\varphi^{-1}(N')$ is a normal subgroup of G.

(8) Let G be a group and N and N' be two normal subgroups of G with the property that $N \cap N' = \{e\}$. Show that for any $n \in N$ and $n' \in N'$, $nn' = n'n$.

(9) Show that $\mathrm{GL}(2, \mathbf{R})/\mathrm{SL}(2, \mathbf{R}) = \mathbf{R}^\times$ using first isomorphism theorem.

(10) In this exercise, we will prove the second isomorphism theorem. Let G be a group and let N and N' be two normal subgroups of G.
- ► Show that $NN' := \{nn' \in G \mid n \in N, \text{ and } n' \in N'\}$ is a subgroup of G.
- ► Show that N' is a normal subgroup of NN'.
- ► Show that $N \cap N'$ is a normal subgroup of N.
- ► Finally, show that $N/N \cap N' \cong NN'/N'$ using first isomorphism theorem. (Hint: Define a homomorphism from N to NN'/N' whose kernel is $N \cap N'$.)