MATH 504
EXERCISES 4

A. ZEYTİN

Unless otherwise stated G is a group.

(1) For each item in the following list, show that the map defines a group action of G on X, determine orbits, the set X/G, the stabilizers and verify orbit stabilizer theorem :

- ▶ $G = (\{\pm 1\}, \cdot)$, $X = \mathbf{R}$,

$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, x) &\mapsto g \bullet x := g \cdot x \end{aligned}$$

- ▶ $G = \mathbf{Z}^2$, $X = \mathbf{R}^2$,

$$\begin{aligned} \bullet \colon G \times X &\to X \\ ((n_1, n_2), (x, y)) &\mapsto g \bullet x := (n_1 + x, n_2 + y) \end{aligned}$$

- ▶ $G = \mathfrak{S}_3$, $X = \mathfrak{S}_3$,

$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, x) &\mapsto g \bullet x := g^{-1} \circ x \circ g \end{aligned}$$

- ▶ $G = \mathfrak{S}_3$, $X = \{$ the set of subgroups of $\mathfrak{S}_3\}$,

$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, H) &\mapsto g \bullet H := g^{-1} H g \end{aligned}$$

- ▶ $G = \mathfrak{S}_4$, $X = \mathfrak{S}_4$,

$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, x) &\mapsto g \bullet x := g \circ x \end{aligned}$$

(2) Let X be a non-empty set admitting an action of a group G.

- ▶ Show that the set $\mathrm{Fix}(G) := \{g \in G \mid g \bullet x = x \text{ for all} x \in X\}$ is a subgroup of G
- ▶ Show that $\mathrm{Fix}(G) = \cap_{x \in X} \mathrm{Stab}(x)$.

(3) Let X be a non-empty set admitting an action of a group G.

(4) Consider the map :

$$\begin{aligned} \bullet \colon \mathrm{GL}(2, \mathbf{R}) \times \mathbf{R}^2 &\to \mathbf{R}^2 \\ (\gamma, (x, y)) &\mapsto \gamma \bullet (x, y) := \gamma \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

- ▶ Show that the above map defines an action of $\mathrm{GL}(2, \mathbf{R})$ on $\mathbf{R}^2$.
- ▶ What is the orbit of $(1, 0)$?
- ▶ What is the stabilizer of $(1, 0)$?