# MATH 504
# EXERCISES 6

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Unless otherwise stated G is a group.

(1) Let G be a group acting on a set X having 5 elements. Given the fact that
- the action og G is faithful, and
- $X/G$ has two equivalence classes, one having 3 elements and another having 2 elements,

what are the possibilities for the group G.

(2) For each of the following groups, find the smallest integer $n$ so that the group acts faithfully on a set of $n$ elements :
- $Q_8$
- $\mathfrak{S}_3$
- $\mathfrak{S}_4$
- $A_3$
- $A_4$

(3) For the following elements, determine whether they are conjugate of not and if yes, find an explicit element conjugating one to the other :
- $\sigma_1 = (1\,5)(2\,5\,6\,4)$, $\sigma_2 = (1\,2\,3)(4\,5)$
- $\sigma_1 = (1\,2\,4)(3\,7)(2\,6\,8\,7)$, $\sigma_2 = (1\,7\,6)(6\,5\,4)(1\,3\,2\,4)$
- $\sigma_1 = (1\,2)(3\,4)(5\,6)$, $\sigma_2 = (1\,2\,3)(4\,5\,6)$

(4) Determine all finite groups having exactly two conjugacy classes.

(5) Let G be a group acting on a set X. Show that G acts faithfully on X if and only if no two distinct elements of G have the same action on each element of X.

(6) Let $\sigma, \tau \in \mathfrak{S}_n$ be arbitrary. If $\sigma$ has cycle decomposition

$$\sigma = (a_1\, a_2\, \cdots\, a_{k_1})(b_1\, b_2\, \ldots\, b_{k_2})\cdots$$

then show that the cycle decomposition of $\tau\sigma\tau^{-1}$ is :

$$\tau\sigma\tau^{-1} = (\tau(a_1)\,\tau(a_2)\,\cdots\,\tau(a_{k_1}))(\tau(b_1)\,\tau(b_2)\,\ldots\,\tau(b_{k_2}))\cdots$$

Deduce that two elements of $\mathfrak{S}_n$ are conjugate if and only if they have the same cycle type, hence the number of conjugacy classes in $\mathfrak{S}_n$ is exactly the number of partitions of $n$.

(7) Let $p$ be a prime number and let G be a $p$-group. Assume that X is a non-empty set admitting an action of G, and that $p \nmid |X|$. Then there is an element $x \in X$ so that $\mathrm{Stab}x = G$. Hint: Use the idea in the proof of the theorem stating that the center of a $p$-group is non-trivial.

(8) Let G be an abelian group. Show that the class equation does not provide any new information on G.

(9) Show that if the group $G/Z(G)$ is cyclic, then G is necessarily abelian.

(10) Verify the class equation for the following groups :
- $\mathfrak{S}_3$
- $\mathfrak{S}_4$
- $A_3$
- $A_4$