MATH 518
EXERCISES 1

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1. We let $N = (0, 1)$, ℓ denote the line with equation $y = 0$, and $\mathcal{C}$ denote the unit circle with equation $x^2 + y^2 = 1$ in $\mathbf{R}^2$. Let $\varphi\colon \ell \to \mathcal{C} \setminus \{N\}$ denote the corresponding stereographic projection.
   - Compute the inverse, $\varphi^{-1}\colon \mathcal{C} \setminus \{N\} \to \ell$, of φ as a function of x and y. Compute the compositions : $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$.
   - Show that $(t, 0)$ is a rational point on ℓ, that is $t \in \mathbf{Q}$, if and only if both x and y coordinates of $\varphi((t, 0))$ are rational. Deduce that there are infinitely many right triangles with integer side lengths.
   - Compute $\varphi((t, 0))$ and $\varphi((\frac{1}{t}, 0))$. Describe the corresponding transformation in $\mathbf{R}^2$ relating these two points on $\mathcal{C}$.
   - Carry out the computations describing the stereographic projection from the "south pole" $S = (0, -1)$ instead of the "north pole" $N = (0, 1)$. Namely, calculate $\psi\colon \ell \to \mathcal{C} \setminus \{S\}$ and $\psi^{-1}\colon \mathcal{C} \setminus \{S\} \to \ell$ explicitly.

2. We let $N = (0, 0, 1)$, $\mathcal{P}$ denote the plane with equation $z = 0$, and $\mathbb{S}$ denote the unit sphere with equation $x^2 + y^2 + z^2 = 1$ in $\mathbf{R}^3$. Let $\varphi\colon \mathcal{P} \to \mathbb{S} \setminus \{N\}$ denote the corresponding stereographic projection.
   - Compute the inverse, $\varphi^{-1}\colon \mathbb{S} \setminus \{N\} \to \mathcal{P}$, of φ as a function of x, y and z. Compute the compositions : $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$.
   - Is it true that if $(s, t, 0)$ is a rational point on $\mathcal{P}$, that is $s, t \in \mathbf{Q}$, if and only if both x, y and z coordinates of $\varphi((s, t, 0))$ are rational.
   - Compute $\varphi((s, t, 0))$ and $\varphi((s, \frac{1}{t}, 0))$. Describe the corresponding transformation in $\mathbf{R}^3$ relating these two points on $\mathbb{S}$.
   - Compute $\varphi((s, t, 0))$ and $\varphi((\dfrac{1}{s}, t, 0))$. Describe the corresponding transformation in $\mathbf{R}^3$ relating these two points on $\mathbb{S}$.
   - Carry out the computations describing the stereographic projection from the "south pole" $S = (0, 0, -1)$ instead of the "north pole" $N = (0, 0, 1)$. Namely, calculate $\psi\colon \mathcal{P} \to \mathbb{S} \setminus \{S\}$ and $\psi^{-1}\colon \mathbb{S} \setminus \{S\} \to \ell$ explicitly.
   - Recall that we have identified the points in the plane $\mathcal{P}$ with complex numbers via $(s, t, 0) \mapsto (z = s + i\,t)$; where $i = \sqrt{-1}$. Let $\mathbb{D}$ denote the unit disk, that is the set $\{z \in \mathbf{C}\colon |z| < 1\}$, in $\mathbf{C}$. Compute $\varphi(\mathbb{D})$. Compare the boundaries of $\mathbb{D}$ in $\mathbf{C}$ and in $\mathbb{S}$.
   - Let $\mathbb{H}$ denote the half plane, that is the set $\{z = s + i\,t \in \mathbf{C}\colon t > 0\}$ in $\mathbf{C}$. Compute $\varphi(\mathbb{H})$.
   - For any $z \in \mathbf{C}$, find the transformation of $\mathbf{R}^3$ relating the points $\varphi(z)$ and $\varphi(-z)$. Compare the boundaries of $\mathbb{H}$ in $\mathbf{C}$ and in $\mathbb{S}$.