MATH 518
EXERCISES 3

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1. (Defining tangent space) Let $P = (X_o, Y_o) \in \mathbf{R}^2 \sim \mathbf{C}$ be a fixed point. We define :

$$\widehat{T_P} := \{\gamma \colon (-\varepsilon, \varepsilon) \to \mathbf{R}^2 \mid \varepsilon > 0,\ \gamma(t) = (x(t), y(t)) \text{ is a smooth curve}^1 \text{ with } \gamma(0) = (x(0), y(0)) = P = (X_o, Y_o)\}.$$

   Two curves say γ_1 and γ_2 in $\widehat{T_P}$ are called equivalent and write $\gamma_1 \sim \gamma_2$, if $\gamma_1'(0) = \gamma_2'(0)$, that is when they have the same *tangent vector* at the point P.
   - Find two different curves that are equivalent to the curve $\gamma(t) = (t, t)$ at $P = (0, 0)$.
   - Find two different curves that are equivalent to the curve $\gamma(t) = (t, t^2)$ at $P = (0, 0)$.
   - Find two different curves that are equivalent to the curve $\gamma(t) = (t + 1, t^2 - 1)$ at $P = (1, -1)$.
   - Find two different curves that are equivalent to the curve $\gamma(t) = (t + 1, t^2 - 1)$ at $P = (0, 0)$.
   - Show that $\sim$ is an equivalence relation.
   - Show that for any element $\gamma \in \widehat{T_P}$, its equivalence class, denoted by $[, gamma]$, contains infinitely many curves.
   - Show that $\widehat{T_P}$ is a two dimensional vector space.

2. We define the *unit disk* in $\mathbf{C}$ as :

$$\mathbb{D} := \{z \in \mathbf{C} \mid |z| < 1\}.$$

   Endow $\mathbb{D}$ with the Poincaré arc-length element $\rho(z) = \dfrac{2}{1 - |z|^2}$. Fix the two points $z_1 = \dfrac{1}{2}(1 + i)$ and $z_2 = \dfrac{1}{2}(1 - i)$.
   - Considering $\mathbb{D}$ as a subset of $\mathbf{R}^2$, calculate the Poincaré arc-length element $\rho(z)$ as a function of x and y.
   - Verify that z_1 and z_2 are elements of $\mathbb{D}$.
   - Parametrize the line segment ℓ joining z_1 to z_2 and calculate its length using the Poincaré arc-length element.
   - Let C be the circle of center 0 and radius $\sqrt{2}/2$. Notice that $z_1, z_2 \in C$. Parametrize the two paths on C joining z_1 to z_2 (one oriented counter clockwise and other clockwise) and calculate their lengths using the Poincaré arc-length element.
   - This time, let D be the circle with radius $\sqrt{5}/2$ and center $3/2 \in \mathbf{C}$. Let γ be the part of D lying inside D. Notice again that the points z_1 and z_2 are points on D. Parametrize the part of D that lies inside $\mathbb{D}$ and calculate the length of γ using the Poincaré arc-length element. Among the three lengths you've calculated, which one is smallest, which one is largest? Can you explain this ordering by looking at the formulation of the arc-length element $\rho(z)$?
   - For a real number $r \in (0, 1)$ let $C_r = \{z \in \mathbf{C} \mid |z| = r\} \subset \mathbb{D}$ be the circle of radius r. Calculate the circumference of C_r using the Poincaré arc-length element.
   - Fix some real number $r \in (0, 1)$ and let $z = i \cdot r$. Show that the length of the line segment joining 0 to $z = i \cdot r$ is $\ln\left(\dfrac{1 + r}{1 - r}\right)$
   - More generally, for any $z_o \in \mathbb{D} \setminus \{0\}$ we let γ_{z_o} be the line joining z_o to $0 \in \mathbb{D}$. Calculate the length of γ_{z_o} using the Poincaré arc-length element.

3. Fix some $z \in \mathbb{H}$ and consider the function $f(z) = \dfrac{z \cdot i + 1}{z + i}$. Verify the following equality

$$\frac{2f'(z)}{1 - |f(z)|^2} = \frac{1}{\operatorname{im}(z)}$$

4. Verify that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is an element of $\mathrm{SL}_(2, \mathbf{R})$ and describe geometrically its action on $\mathbb{H}$.

5. Let $a, b \in \mathbf{R}$ be two arbitrary real numbers, with $a > 0$. Show that all transformations of the form $z \mapsto az + b$ can be realized as an element of $\mathrm{SL}_(2, \mathbf{R})$.