MATH 518
EXERCISES 4

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1. In this long exercise, we will understand more the action of $\mathrm{GL}(2, \mathbf{C})$ (that is the *group* of 2×2 matrices with non-zero determinant) on $\mathbf{C} \cup \{\infty\}$ that we defined in this week's quiz; namely, we defined :

$$\bullet\colon \mathrm{GL}(2,\mathbf{C}) \times \mathbf{C}\cup\{\infty\} \to \mathbf{C}\cup\{\infty\}$$
$$\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}, z\right) \mapsto \frac{pz+q}{rz+s}.$$

   - ▶ Show that given any three *distinct* complex numbers z_1, z_2, z_3 in $\mathbf{C}$, the map
   $$m(z) := \frac{(z_1 - z)}{(z_3 - z)} \frac{(z_3 - z_2)}{(z_1 - z_2)}$$
   satisfies :
     - $m(z_1) = 0$
     - $m(z_2) = 1$
     - $m(z_3) = \infty$.
   - ▶ Extend the previous exercise to $\mathbf{C}\cup\{\infty\}$, that is explain what to do if one of z_i is equal to ∞.
   - ▶ Given any element $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}(2,\mathbf{C})$, and any non-zero complex number α, show that the action of M and αM on $\mathbf{C}\cup\{\infty\}$ are the same. In particular, given any map $\dfrac{pz+q}{rz+s}$ from $\mathbf{C}\cup\{\infty\}$ to itself with $ps - qr \neq 0$, there is at least one element of $\mathrm{SL}(2,\mathbf{C})$ with the same action on $\mathbf{C}\cup\{\infty\}$.
   - ▶ Conclude that given two triples of distinct elements of $\mathbf{C}\cup\{\infty\}$, say z_1, z_2, z_3 and w_1, w_2, w_3, there is at least one element of $\mathrm{SL}(2,\mathbf{C})$ sending z_i to w_i for $i = 1, 2, 3$. Technically speaking, we say that $\mathrm{SL}(2,\mathbf{C})$ acts *triply transitively* on $\mathbf{C}\cup\{\infty\}$.

2. Given four *distinct* complex numbers in $\mathbf{C}$ we define their *cross ratio*, denoted $[z_1, z_2, z_3, z_4]$, as :
$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_4)}{(z_3 - z_4)} \frac{(z_3 - z_2)}{(z_1 - z_2)}.$$
   - ▶ Calculate the following cross ratios : $[1, 2, 3, 4]$, $[2, 3, 4, 1]$, $[3, 4, 1, 2]$ and $[4, 1, 2, 3]$.
   - ▶ Calculate the following cross ratios : $[\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}]$, $[\sqrt{-2}, \sqrt{-3}, \sqrt{-4}, \sqrt{-1}]$, $[\sqrt{-3}, \sqrt{-4}, \sqrt{-1}, \sqrt{-2}]$ and $[\sqrt{-4}, \sqrt{-1}, \sqrt{-2}, \sqrt{-3}]$.
   - ▶ Calculate the following cross ratios : $[1, -1, \sqrt{-1}, -\sqrt{-1}]$, $[-1, \sqrt{-1}, -\sqrt{-1}, 1]$, $[\sqrt{-1}, -\sqrt{-1}, 1, -1]$ and $[-\sqrt{-1}, 1, -1, \sqrt{-1}]$.
   - ▶ Extend the cross ratio from $\mathbf{C}$ to $\mathbf{C}\cup\{\infty\}$ by way of taking limits. More precisely, if $z_1 = \infty$ then we define
   $$[\infty, z_2, z_3, z_4] := \lim_{z\to\infty} \frac{(z - z_4)}{(z_3 - z_4)} \frac{(z_3 - z_2)}{(z - z_2)}$$
   We define $[z_1, \infty, z_3, z_4]$, $[z_1, z_2, \infty, z_4]$ and $[z_1, z_2, z_3, \infty]$ similarly. Calculate $[-1, 0, 1, \infty]$, $[\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \infty]$ and $[1, \sqrt{-1}, -1, \infty]$.
   - ▶ Let $M \in \mathrm{SL}(2,\mathbf{C})$ be an arbitrary element. Show that
   $$[z_1, z_2, z_3, z_4] = [M \bullet z_1, M \bullet z_2, M \bullet z_3, M \bullet z_4].$$
   In technical terms, we say that cross ratio is *invariant* under the action of $\mathrm{SL}(2,\mathbf{C})$.
   - ▶ Compute $[\infty, 0, 1, z]$; where $z \in \mathbf{C} \setminus \{0, 1\}$. Deduce that $[\infty, 0, 1, z] \in \mathbf{R}$ if and only if $z \in \mathbf{R}$.
   - ▶ Conclude that the points z_1, z_2, z_3, z_4 lie on a circle or on a line in $\mathbf{C}\cup\{\infty\}$ if and only if $[z_1, z_2, z_3, z_4] \in \mathbf{R}$.

3. Decide whether the following quadruple of points lie on a circle :
   - ▶ $1 + \sqrt{-1}, 1 - \sqrt{-1}, 0, \sqrt{-4}$
   - ▶ $2 - \sqrt{-1}, 2 + \sqrt{-1}, 2 + \sqrt{-2}, 2 - \sqrt{-2}$

4. Let $z_1 = x_1 + \sqrt{-1}y_1$ and $z_2 = x_2 + \sqrt{-1}y_2$ be two points in $\mathbb{H}$ with $\mathrm{re}(z_1) \neq \mathrm{re}(z_2)$. Our aim is to find the circle that is perpendicular to $\partial\mathbb{H} = \mathbf{R} \cup \{\infty\}$.
   - We let $\ell(z_1, z_2)$ be the Euclidean line joining z_1 to z_2, and z_3 be the midpoint of z_1 and z_2. Find an equation of the line perpendicular to $\ell(z_1, z_2)$ passing through z_3. Call this line $\overline{\ell(z_1, z_2)}$.
   - Find the intersection point of $\overline{\ell(z_1, z_2)}$ with $\partial\mathbb{H}$. Call this point z_o. Convince yourselves that this should be the center of the circle we are looking for.
   - Verify that $z_o \in \mathbf{R}$.
   - Verify that the Euclidean distance between z_1 and z_o is equal to that of z_2 and z_o. Call this distance r.
   - Using cross ratio that we defined in the previous exercise, verify that $z_o - r, z_1, z_2, z_o + r$ lie on a circle.

5. Compute the hyperbolic distance between the following points of $\mathbb{H}$
   - $z_1 = \sqrt{-1}, z_2 = 1 + \sqrt{-1}$
   - $z_1 = 1 + \sqrt{-1}, z_2 = 2 + \sqrt{-1}$
   - $z_1 = \sqrt{-1}2, z_2 = 1 + \sqrt{-1}2$
   - $z_1 = 3 + \sqrt{-1}, z_2 = 3 + \sqrt{-1}2$
   - $z_1 = 1 + \sqrt{-1}\frac{1}{2}, z_2 = \sqrt{-1}\frac{1}{2}$

6. Endow $\mathbf{R}$ with the usual Euclidean metric ($d(x, y) = |x - y|$). Which of the following transformations are isometries :
   - $f(x) = \dfrac{x-3}{4}$
   - $f(x) = x^3$
   - $f(x) = -2x$
   - $f(x) = x - 2$

7. Endow $\mathbf{R}^2$ with the usual Euclidean metric ($d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$). Which of the following transformations are isometries :
   - $f(x, y) = (x + 1, y - 3)$
   - $f(x) = (3x, 2y)$

8. Let (X, d) be a metric space and $m \colon X \to X$ and $g \colon X \to X$ be two self-isometries of X, in particular f and g are both surjective. Show the following fundamental properties :
   - f is injective, and therefore a bijection.
   - f is continuous.
   - f^{-1} is an isometry.
   - $f \circ g$ is an isometry. Deduce that the set of all isometries of a metric space (X, d), denoted $\mathrm{Isom}(X)$ is a group under composition.