MATH 518
EXERCISES 5

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1. Compute, if possible, the lengths of paths joining the following points in $\mathbb{D}$ using the Poincaré metric on $\mathbb{D}$. Then calculate, if possible, lengths of the paths obtained by mapping the path to upper half plane by Cayley transform. Finally, compute the actual distance using the hyperbolic metric on $\mathbb{H}$ by way of finding an appropriate transformation that maps these points to the imaginary axis in $\mathbb{H}$:
   - the straight line segment joining 0 to $\frac{1}{2}$
   - the straight line segment joining 0 to $\frac{\sqrt{1}+\sqrt{-1}}{2}$
   - the straight line segment joining $\frac{1}{2}$ to $\frac{1+\sqrt{-1}}{2}$
   - the straight line segment joining $\frac{1+\sqrt{-1}}{2}$ to $\frac{12\sqrt{-1}}{2}$
   - the short arc of the circle with center 0 and radius $\frac{\sqrt{2}}{2}$ joining $\frac{1+\sqrt{-1}}{2}$ to $\frac{12\sqrt{-1}}{2}$
   - the long arc of the circle with center 0 and radius $\frac{\sqrt{2}}{2}$ joining $\frac{1+\sqrt{-1}}{2}$ to $\frac{12\sqrt{-1}}{2}$
2. Show that the Poincaré metric on $\mathbb{D}$ is invariant under the group $\mathrm{SU}(1,1)$ by direct computation.
3. In this exercise, we accept the fact that Euclidean circles in $\mathbb{D}$ and in $\mathbb{H}$ are hyperbolic circles, too. That is, if C is a Euclidean circle of the form
$$C = \{z = x + y\sqrt{-1} : (x-a)^2 + (y-b)^2 = r^2\} = \{z : \left|z - (a + b\sqrt{-1})\right|^2 = r^2\}$$
then, there is a point $z_o \in \mathbb{D}$ or $z_o \in \mathbb{H}$ (possibly different from $a + b\sqrt{-1}$) so that
$$C = \{z \in \mathbb{D} : d_{\text{Poincaré}}(z, z_o) = (r')^2\} \text{ or } C = \{z \in \mathbb{H} : d_{\text{hyperbolic}}(z, z_o) = (r')^2\}$$
for some $r' > 0$ (possibly different from r). For the following Euclidean circles, find the Poincaré or hyperbolic centers and radii
   - circle of center 0 and radius $1/2$in $\mathbb{D}$,
   - circle of center $\frac{1}{5} - \frac{1}{4}\sqrt{-1}$ and radius $1/10$in $\mathbb{D}$,
   - circle of center $1 + 3i$ and radius 1in $\mathbb{H}$,
4. More generally, show that if C is a Euclidean circle in $\mathbb{H}$ with Euclidean center $a + b\sqrt{-1}$ and Euclidean radius r, then its hyperbolic center must be $a + \sqrt{r^2 - b^2}$. What is an analoguous statement for Euclidean circles in $\mathbb{D}$.