MATH 518
EXERCISES 8

A. ZEYTİN

1. Compute (hyperbolic or Poincaré) areas of the triangles whose vertices are given as :
   - $v_1 = 2, v_2 = 3 + \sqrt{-1}\, v_3 = \infty$ in $\mathbb{H}$
   - $v_1 = 0, v_2 = 1, v_3 = \dfrac{3+\sqrt{-3}}{2}$ in $\mathbb{H}$
   - $v_1 = 1 + \sqrt{-3}, v_2 = 3 + \sqrt{-3}, v_3 = 2$ in $\mathbb{H}$
   - $v_1 = 2, v_2 = 3 + \sqrt{-3}, v_3 = 4$ in $\mathbb{H}$
   - $v_1 = 0, v_2 = 1, v_3 = \sqrt{-1}$ in $\mathbb{D}$
2. Prove that
   - the hyperbolic area of a hypebolic quadrtilateral is $2\pi - \sum_{i=1}^{4}$; where α_i are the internal angles of the quadrilateral.
   - the hyperbolic area of a hyperbolic $n-$gon, denoted by P_n with internal angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ is given by the formula :
   $$\mathrm{Area}(P_n) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$
3. Let G be a group acting on a non-empty set X. Show that
   - for every fixed $g \in G$, the map :
   $$\begin{aligned} \iota_g &: X \to X \\ x &\mapsto \iota_g(x) := g \bullet x \end{aligned}$$
   is a bijection and therefore an element of the set of all permutations of X, denoted by $\mathfrak{S}(X)$. <u>Hint:</u> What is ι_g^{-1}?
   - Deduce that the map
   $$\begin{aligned} \varphi &: G \to \mathfrak{S}(X) \\ g &\mapsto \iota_g \end{aligned}$$
   is a group homomorphism. This is called the *permutation representation* of the action of G on X.
   - Recall that the action of G on X is called faithful (or effective) if $g \bullet x = x$ implies $g = e_G$. Show that faithfulness of a group action is equivalent to φ being injective. <u>Hint:</u> Study elements of $\ker(\varphi)$.
   - Recall that the action of G on X is called transitive if for and $x, y \in X$, there is a $g \in G$ with $y = g \bullet x$. Show that an action is transitive if and only if $|G/X| = 1$.
4. Given an action of a group G on a non-empty set X, for any element $x \in X$ we define the *stabilizer* of x in G as :
   $$G_x = \mathrm{Stab}_G(x) := \{g \in G \,|\, g \bullet x = x\}$$
   - Show that G_x is a subgroup of G.
   - Give an example to show that G_x need not be a normal subgroup of G.
   - Let $x, y \in X$ we two elements so that $y = g \bullet x$. Show that $G_y = gG_xg^{-1}$.
   - For any $x \in X$ show that the map
   $$\begin{aligned} \iota_x &: [x] \to G/G_x \\ g \bullet x &\mapsto gG_x \end{aligned}$$
   is a well-defined bijection.
   - Deduce the orbit-stabilizer theorem :
   $$|[x]| = [G : G_x]$$

5. Let G be a group acting on a non-empty set X. We say that the action is *free* (or G acts freely on X) if for all $x \in X$ and $g \in G \setminus \{e_G\}$ we have $g \bullet x \neq x$. Show that G acts freely if and only if $G_x = \{e_G\}$ for all $x \in X$.

6. We let X be the set of all 2×2 symmetric positive definite matrices; that is

$$X = \left\{ q = \begin{pmatrix} a & b \\ b & c \end{pmatrix} | a, b, c \in \mathbf{R},\ (X\,Y)q(X\,Y)^t \geq 0 \text{ for all } (X\,Y) \in \mathbf{R}^2 \right\}$$

Here, $\cdot^t$ denotes the transpose.

- Give a criterion for positive definiteness of $q \in X$ in terms of a and $\det(q)$.
- Show that the set X admits an action of $\mathrm{GL}(2, \mathbf{R})$ defined as follows :

$$\mathrm{GL}(2, \mathbf{R}) \times X \to X$$
$$(A, q) \mapsto A \bullet q := AqA^t$$

- Show that this action is transitive. <u>Hint:</u> Use diagonalizability of symmetric matrices.
- Is this action faithful?
- Is this action free?
- Generalize this to square matrices of arbitrary size.

7. On the set $X = \mathbf{R}^n \setminus \{\mathbf{0}\}$ we define an action of the multiplicative group $\mathbf{R}^\times = \mathbf{R} \setminus \{0\}$ as :

$$(\lambda, \mathbf{v}) \mapsto \lambda \bullet \mathbf{v} := \lambda\mathbf{v}$$

- Show that this defines an action of $\mathbf{R}^\times$ on X.
- Is this action transitive?
- Is this action faithful?
- Is this action free?
- Let $L = \{\ell \subset \mathbf{R}^n \,|\, \ell \text{ is a line through origin in } \mathbf{R}^n\}$, that is the set of lines passing through origin in $\mathbf{R}^n$. Establish a bijection between $X/\mathbf{R}^\times$ and L. <u>Hint:</u> Start with the particular case where $n = 2$ and $n = 3$.

8. Consider the additive group $G = \mathbf{Z} \times \mathbf{Z}$. Consider the following map :

$$\bullet : G \times \mathbf{C} \to \mathbf{C}$$
$$((n, m), x + y\sqrt{-1}) \mapsto (n, m) \bullet (x + y\sqrt{-1}) := x + n + (y + m)\sqrt{-1}$$

- Show that this is indeed an action of G on $\mathbf{C}$.
- Is this action transitive?
- Is this action faithful?
- Is this action free?
- Establish a bijection between $\mathbf{C}/G$ and $S^1 \times S^1$; where $S^1 = \{z \in \mathbf{C} \,|\, |z| = 1\}$. <u>Hint:</u> It might be easier to first establish a bijection between $\mathbf{R}/\mathbf{Z}$ and S^1.