MATH 504
EXERCISES 10

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Every ring R is assumed to be commutative with 1_R.

(1) Show that if R is a principal ideal domain, then every non-zero prime ideal is maximal.

(2) Show that if k is a field then $k[X]$ is a principal ideal domain. Is $k[X, Y]$ a principal ideal domain?

(3) Construct fields of orders $4, 8, 9, 25$ and 27.

(4) Show that every abelian group is a $\mathbf{Z}$ module.

(5) Let M be an R module and N a subset of M. Show that N is an R-submodule of M if and only if :
- $N \neq \emptyset$, and
- $m + rm' \in N$ for all $r \in R$ and $m, m' \in N$.

(6) Prove that $R^\times$ (i.e. the unit group of R) and satisfy the axioms of a group action, i.e. $R^\times$ acts on M as a group.

(7) Let R be a ring. Consider R^n as an R module (as discussed in the lecture). Decide whether the following sets are submodules of M :
- $\{(x_1, x_2, \ldots, x_n) \,|\, x_i \in I_i\}$; where I_i is an ideal of R for each $i = 1, 2, \ldots, n$
- $\{(x_1, x_2, \ldots, x_n) \,|\, x_i \in R_i\}$; where R_i is a sub-ring of R for each $i = 1, 2, \ldots, n$
- $\{(x_1, x_2, \ldots, x_n) \,|\, x_1 + x_2 + \ldots + x_n = 0\}$
- $\{(x_1, x_2, \ldots, x_n) \,|\, x_1 + x_2 + \ldots + x_n = 1\}$
- $\{(x_1, x_2, \ldots, x_n) \,|\, a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0\}$ for some fixed $a_i \in R$
- $\{(x_1, x_2, \ldots, x_n) \,|\, a_1x_1 + a_2x_2 + \ldots + a_nx_n = 1\}$

(8) Show that the intersection of any non-empty collection of submodules of an R module M is again a submodule. Is the union of two submodules again a submodule?

(9) Let M be an R-module. An element $\in M$ is called a torsion element of $rm = 0$ for some $r \in R$. The set of all torsion elements in M is denoted by : $\mathrm{Tor}(M)$. Show that
- $\mathrm{Tor}(M)$ is not necessarily a submodule of M.
- $\mathrm{Tor}(M)$ is a submodule of M if R is an integral domain.
- if R is not an integral domain, show that any non-zero R-module M has non-trivial torsion.
- An R-module M is called *torsion* if $M = \mathrm{Tor}(M)$. Show that any finite abelian group is a torsion $\mathbf{Z}$-module.
- Is the converse of the previous exercise true, that is, is every torsion module a finite abelian group?

(10) For any R module M, the set $\mathrm{Hom}_R(M, M)$ is called the endomorphism ring of M and denoted usually by $\mathrm{End}(M)$.
- Show that $\mathrm{End}(M)$ is a ring.
- Show that for any fixed element $r \in R$, the map $p_r(m) := rm$ defines an element of $\mathrm{End}(M)$.

(11) Determine all $\mathbf{Z}$ module homomorphisms
- from $\mathbf{Z}/12\mathbf{Z}$ to $\mathbf{Z}/18\mathbf{Z}$
- from $\mathbf{Z}/30\mathbf{Z}$ to $\mathbf{Z}/21\mathbf{Z}$
- from $\mathbf{Z}/12\mathbf{Z}$ to $\mathbf{Z}/12\mathbf{Z}$

(12) Let M be an R-module and let N be an R-submodule of M. Show that if both N and M/N are finitely generated, then so is M.

(13) Let M, M' and N be three R-modules. Prove that the following R-module isomorphisms :
- $\mathrm{Hom}_R(M \oplus M', N) \cong \mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(M', N)$
- $\mathrm{Hom}_R(N, M \oplus M') \cong \mathrm{Hom}_R(N, M) \oplus \mathrm{Hom}_R(N, M')$

(14) Show that the direct sum of two free R-modules is free.

(15) Show that if $M \subseteq M' \subseteq M''$ are three R-modules then The modules $(M/M'')/(M'/M'')$ and M/M' are isomorphic.