MATH 504
EXERCISES 10

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Every ring R is assumed to be commutative with 1_R.

(1) Let M be a finitely generated R module and I be an ideal of R. Suppose that $\varphi\colon M \to M$ is an R-module homomorphism of M with the property that $\mathrm{im}(\varphi) \subseteq I \cdot M$. Show that there are elements $\alpha_0, \alpha_1, \dots, \alpha_n \in I$ so that
$$\alpha_0 + \alpha_1 \varphi + \dots + \alpha_n \varphi^n = 0_M$$
as an R-module homomorphism from M to M. Hint: As M is finitely generated, you may choose a set of generators, say x_i. Write $\varphi(x_i)$ in terms of these generators and form a "matrix". Then use the adjoint of this "matrix".

(2) Using the previous exercise prove a version of chinese remainder theorem, that is if I is an ideal of R so that $I \cdot M = M$; where M is a finitely generated R-module, then there exists some $x \in R$ so that $x \equiv 1 \pmod{I}$ so that $xM = 0$.

(3) (Splitting exact sequences) Suppose that there are two exact sequences of the form :
$$M'' \to M \to M' \to 0 \quad \text{and} \quad M' \to N \to N''.$$
Show that one can obtain the following exact sequence :
$$M'' \to M \to N \to N''.$$

(4) (Gluing exact sequences) Say we are given an exact sequence of the form :
$$M'' \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} N''.$$
Show that the sequences
$$M'' \xrightarrow{f} M \xrightarrow{g} K \to 0 \quad \text{and} \quad K \xrightarrow{\iota} N \xrightarrow{h} N'';$$
where $K = \mathrm{im}(g) = \ker(h)$.

(5) (Splitting lemma) Let $0 \to M'' \xrightarrow{f} M \xrightarrow{g} M' \to 0$ be a short exact sequence. Show that the following statements are equivalent :
- $M \cong M'' \oplus M'$
- f has a left inverse, that is, there is an R-module homomorphism $\varphi\colon M \to M''$ so that $\varphi \circ f = \mathrm{id}_{M''}$
- g has a right inverse, that is, there is an R-module homomorphism $\psi\colon M' \to M$ so that $f \circ \psi = \mathrm{id}_{M'}$

If one of the above equivalent conditions hold, then the corresponding sequence is called *split exact*.

(6) Fix a field K. Show that every short exact sequence $0 \to V'' \to V \to V' \to 0$ of K-vector spaces is split exact.

(7) Given the fact that the sequence :
$$0 \to \mathbf{Z} \xrightarrow{f} \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus (\mathbf{Z}/2\mathbf{Z}) \to 0$$
is exact, what can you say about the $\mathbf{Z}$-module homomorphism f?