MATH 504
EXERCISES 12

A. ZEYTİN

Every ring R is assumed to be commutative with 1_R.

(1) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Show that if M' and M'' are both finitely generated, then so is M.

(2) Prove that $V \otimes W$ is isomorphic to $\mathrm{Hom}_V(W)$. Hint: Choose bases! This in particular means that the isomorphism is not natural!

(3) Verify the following isomorphisms :
- $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$
- $\mathbf{Z}[X]/(f(X)) \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z} \cong (\mathbf{Z}/p\mathbf{Z})[X]/(f(X))$; where p is a prime number and $f(X) \in \mathbf{Z}[X]$ is an irreducible polynomial.
- $R[X] \otimes R[Y] \cong R[X, Y]$
- $R \otimes_R M$; where M is an R-module
- $\mathbf{Q} \otimes A \cong 0$; where is any *finite* abelian group.

(4) Let R be a ring and let M be a free R-module of rank n, such as an n-dimensional vector space over a field. Choose a basis $\{e_1, \ldots, e_n\}$ of M as an R-module. Show that
- $e_1 \otimes e_1 + e_2 \otimes e_2 \in M \otimes_R M$
- $e_1 \otimes e_2 + e_2 \otimes e_1 \in M \otimes_R M$

cannot be written as $\alpha \otimes \beta$ for any $\alpha,\ \beta \in M$.

(5) Show that for two R-modules M and N the tensor product $M \otimes N = 0$ if and only if every bilinear map from $M \times N$ to any R-module P is identically 0.