MATH 504
EXERCISES 2

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Unless otherwise stated G is a group.

(1) Show that the intersection of any collection (finite or infinite, countable or uncountable) of subgroups of a group G is again a subgroup. Deduce that for an arbitrary non-empty subset $X \subset G$ the set of all subgroups containing X, denoted $\langle X \rangle$ is a subgroup of G called the subgroup generated by X. Deduce further that if X is itself a subgroup, then the subgroup generated by X is itself, that is show that $\langle X \rangle = X$ when X is a subgroup.
- ► Determine the subgroup generated by $X = \{(1\,2), (1\,3)\}$ in $\mathfrak{S}_3$.
- ► Determine the subgroup generated by $X = \{(1\,2), (1\,3)\}$ in $\mathfrak{S}_4$.
- ► If $X = \{g\}$, then show that the subgroup $\langle X \rangle = \{g^n \mid n \in \mathbf{Z}\}$. Such subgroups are called *cyclic*. If $G = \langle \{g\} \rangle$ for some $g \in G$ then G is called cyclic.
- ► Give an example of a finite cyclic subgroup.
- ► Give an example of an infinite cyclic subgroup.
- ► Let G be a group and $a, b \in G$ be two elements. Try to list all the elements of the subgroup $\langle \{a, b\} \rangle$ if $a^2 = e = b^3$.
- ► Let G be a group and $a, b \in G$ be two elements. Try to list all the elements of the subgroup $\langle \{a, b\} \rangle$ if $a^2 = e = b^3$ and $ab = ba$.

(2) Find a relation which is
- ► reflexive but neither symmetric nor transitive.
- ► symmetric but neither reflexive nor transitive.
- ► transitive but neither symmetric nor reflexive.
- ► both symmetric and reflexive but not transitive.
- ► both reflexive and transitive but not symmetric.
- ► both symmetric and transitive but not reflexive.

(3) On $\mathbf{Z} \setminus \{0\}$, we define
$$R = \{(m, n) \in (\mathbf{Z} \setminus \{0\})^2 \mid mn > 0\}.$$
- ► Sow that R is an equivalence relation.
- ► Determine all the equivalence classes and the partition of $\mathbf{Z} \setminus \{0\}$ determined by them.

(4) On $\mathbf{R}^2$, we define
$$(p, q) \sim_R (s, t) :\Leftrightarrow \frac{q - t}{s - t} = 2.$$
- ► Show that $\sim_R$ is an equivalence relation.
- ► Give a geometric description of equivalence classes.

(5) Let X be a non-empty set and T be a fixed subset of X. On $\mathcal{P}(X)$ we define :
$$A \sim_R B :\Leftrightarrow A \cap T = B \cap T.$$
- ► Show in general that $\sim_R$ is an equivalence relation.
- ► Determine all the equivalence classes when $T = \emptyset$.
- ► Determine all the equivalence classes when $T = X$.
- ► Set $X = \{1, 2, 3, 4\}$ and $T = \{1, 3\}$. Describe the equivalence classes and deduce the corresponding partition of $\mathcal{P}(X)$.

(6) Decide which of the following relations of $\mathbf{R}^2$ is an equivalence relation :
- ► $(x_1, y_1) \sim_R (x_2, y_2) :\Leftrightarrow x_1^2 - x_2^2 = y_1 - y_2$
- ► $(x_1, y_1) \sim_R (x_2, y_2) :\Leftrightarrow x_1^2 - x_2^2 = y_1^2 - y_2^2$
- ► $(x_1, y_1) \sim_R (x_2, y_2) :\Leftrightarrow x_1 + x_2 = y_1 + y_2$
- ► $(x_1, y_1) \sim_R (x_2, y_2) :\Leftrightarrow x_1 y_1 = x_2 y_2$

(7) Let $\varphi\colon G \to G'$ be a group homomorphism and let e' denote the identity element of G'.
- Show that the set $\ker(\varphi) := \{g \in G\colon \varphi(g) = e'\}$ is a normal subgroup of G. This subgroup is called the kernel of φ.
- Show that the set $\mathrm{im}(()\varphi) := \{g' \in G'\colon g' = \varphi(g) \text{ for some } g \in G\}$ is a subgroup of G'. This subgroup is called the image of φ.
- Show by an explicit example that $\mathrm{im}((\varphi))$ need not be a normal subgroup.

(8) Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?
- $$\begin{aligned}\varphi\colon \mathbf{R}^\times &\to \mathrm{GL}(2,\mathbf{R})\\ x &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\end{aligned}$$
- $$\begin{aligned}\varphi\colon \mathbf{R} &\to \mathrm{GL}(2,\mathbf{R})\\ x &\mapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\end{aligned}$$
- $$\begin{aligned}\varphi\colon \mathrm{GL}(2,\mathbf{R}) &\to \mathbf{R}\\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto p + s\end{aligned}$$
- $$\begin{aligned}\varphi\colon \mathrm{GL}(2,\mathbf{R}) &\to \mathbf{R}\\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} &\mapsto q + r\end{aligned}$$
- $$\begin{aligned}\varphi\colon \mathbf{Z} &\to \mathbf{Z}\\ n &\mapsto 504n\end{aligned}$$

(9) Let $n \in \mathbf{N}$ be an integer with $n > 1$. Show that if G is an abelian group, then the map
$$\begin{aligned}\varphi_n\colon G &\to G\\ g &\mapsto g^n\end{aligned}$$
is a group homomorphism. Show further that φ_n need not be a group homomorphism in general.

(10) Show that if G is an abelian group and $\varphi\colon G \to G'$ is a group homomorphism, then $\mathrm{im}(()\varphi)$ is an abelian subgroup of G'.

(11) Let G be a finite group, that is $|G| = n \in \mathbf{N}$. Show that there is an integer m so that $g^m = e$ for all $g \in G$. Show that one may take $m = n$, that is, show that for all $g \in G$ we have $g^{|G|} = e$. Give an example where one may choose $m < |G|$.

(12) Let G be a group, H be a subgroup and N be a normal subgroup of G. Show that $NH = \{nh \in G \mid n \in N,\ h \in H\}$ is a subgroup. Show, by an example, that this fails when N is not normal.

(13) Show that the intersection of two normal subgroups is again a normal subgroup.

(14) Let $\varphi\colon G \to G'$ be a group homomorphism. Show that φ is one-to-one if and only if $\ker(\varphi) = \{e\}$.

(15) Let $\varphi\colon G \to G'$ be a group homomorphism. Define $g_1 \sim g_2$ when $\varphi(g_1) = \varphi(g_2)$.
- Show that the mentioned relation is an equivalence relation.
- Describe the equivalence classes of this relation.