MATH 504
EXERCISES 3

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Unless otherwise stated G and G' are groups.

(1) Show that if G is a cyclic group, that is $G = \langle g \rangle$ for some $g \in G$, then for any group G', any homomorphism $\varphi\colon G \to G'$ is determined by $\varphi(g)$. More generally, show that if G is a group generated by $g_1, g_2, \dots, g_n$ then any homomorphism $\varphi\colon G \to G'$ is determined by $\varphi(g_1), \varphi(g_2). \dots, \varphi(g_n)$.

(2) Determine all homomorphisms from :
- ▶ $\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$
- ▶ $\mathbf{Z}/7\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$
- ▶ $\mathbf{Z}/8 \to \mathbf{Z}/7\mathbf{Z}$
- ▶ $\mathbf{Z}/14\mathbf{Z} \to \mathbf{Z}/7\mathbf{Z}$
- ▶ $\mathbf{Z}/7\mathbf{Z} \to \mathbf{Z}/14\mathbf{Z}$
- ▶ $\mathfrak{S}_3 \to \mathbf{Z}$
- ▶ $\mathbf{Z} \to \mathfrak{S}_3$

(3) Let m and n be two relatively prime numbers. Show that there is no non-trivial group homomorphism from $\mathbf{Z}/m\mathbf{Z}$ to $\mathbf{Z}/n\mathbf{Z}$.

(4) Let $\varphi\colon G \to G'$ be a group homomorphism.
- ▶ Show that for any $g \in G$ the order of $\varphi(g)$ divides the order of g, that is $\mathrm{ord}(\varphi(g)) | \mathrm{ord}(g)$.
- ▶ Show that if φ is an isomorphism then its inverse $\varphi^{-1}\colon G' \to G$ is also an isomorphism.
- ▶ Deduce that if φ is an isomorphism, then $\mathrm{ord}(g) = \mathrm{ord}(\varphi(g))$.

(5) An automorphism of a group G is defined as an isomorphism $\varphi\colon G \to G$. Determine all automorphisms of the following groups :
- ▶ $\mathbf{Z}/4\mathbf{Z}$
- ▶ $\mathbf{Z}/5\mathbf{Z}$
- ▶ $\mathbf{Z}/6\mathbf{Z}$

(6) Let $G = \mathfrak{S}_n$ be the symmetric group on $\{1, 2, \dots, n\}$.
- ▶ A transposition is a cycle of length 2, i.e. elements of the form (a, b) for $a, b \in \{1, 2, \dots, n\}$. Show that $\mathfrak{S}_n$ is generated by transpositions, that is any $\sigma \in \mathfrak{S}_n$ can be written as a product of transpositions.
- ▶ Notice that writing an element as a product of transpositions is not unique. However, for any element $\sigma \in \mathfrak{S}_n$ whenever

$$\begin{aligned}\sigma &= (a_1, b_1)(a_2, b_2)\dots(a_k, b_k)\\ &= (a_1, b_1)(a_2, b_2)\dots(a_l, b_l)\end{aligned}$$

  show that $(-1)^k = (-1)^l$; that is the parity of k (or l) is well-defined.
- ▶ Deduce that the map $\mathrm{sign}\colon \mathfrak{S}_n \to \{\pm 1\}$ sending σ to $(-1)^k$; where k is the number of transpositions used in writing σ as their product is a group homomorphism.
- ▶ Define A_n to be $\ker(\mathrm{sign})$, that is A_n is the subgroup of even permutations. Deduce that A_n is a normal subgroup of $\mathfrak{S}_n$

(7) Let G be a group, H and K be subgroups of G.
- ▶ Show that $H \times K = \{(h, k) \,|\, h \in H, k \in K\}$ is a group under componentwise multiplication, that is $(h, k) * (h', k') = (hh', kk')$.
- ▶ Show that the subset $A = \{(h, e) \,|\, h \in H\}$ is a normal subgroup of $H \times K$.
- ▶ Show that the subset $B = \{(e, k) \,|\, k \in K\}$ is a normal subgroup of $H \times K$.
- ▶ Find a group G' and establish a homomorphism whose kernel is A. Use first isomorphism theorem to deduce that $K \cong (H \times K)/A$

- Find a group G'' and establish a homomorphism whose kernel is B. Use first isomorphism theorem to deduce that $H \cong (H \times K)/B$

(8) Let $\varphi\colon G \to G'$ be a group epimorphism and N be a normal subgroup of G. Show that $\varphi(N)$ is a normal subgroup of G'. Show by an example that the claim fails to hold if we do not assume φ to be an epimorphism.

(9) Let $\varphi\colon G \to G'$ be a group homomorphism and N' be a normal subgroup of G'. Show that $\varphi^{-1}(N')$ is a normal subgroup of G.

(10) Let G be a group and N and N' be two normal subgroups of G with the property that $N \cap N' = \{e\}$. Show that for any $n \in N$ and $n' \in N'$, $nn' = n'n$.

(11) Show that $GL(2, \mathbf{R})/SL_{2,\mathbf{R}}(=)\mathbf{R}^{\times}$ using first isomorphism theorem.

(12) In this exercise, we will prove the second isomorphism theorem. Let G be a group and let N and N' be two normal subgroups of G.
- Show that $NN' := \{nn' \in G \,|\, n \in N, \text{ and } n' \in N'\}$ is a subgroup of G.
- Show that N' is a normal subgroup of NN'.
- Show that $N \cap N'$ is a normal subgroup of N.
- Finally, show that $N/N \cap N' \cong NN'/N'$ using first isomorphism theorem. (Hint: Define a homomorphism from N to NN'/N' whose kernel is $N \cap N'$.)