MATH 504
EXERCISES 4

A. ZEYTİN

Unless otherwise stated G and G' are groups and X is a non-empty set admitting an action of G.

(1) For each item in the following list, show that the map defines a group action of G on X, determine orbits, the set X/G, the stabilizers and verify orbit stabilizer theorem :

- $G = (\{\pm 1\}, \cdot)$, $X = \mathbf{R}$,
$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, x) &\mapsto g \bullet x := g \cdot x \end{aligned}$$
- $G = \mathbf{Z}^2$, $X = \mathbf{R}^2$,
$$\begin{aligned} \bullet \colon G \times X &\to X \\ ((n_1, n_2), (x, y)) &\mapsto g \bullet x := (n_1 + x, n_2 + y) \end{aligned}$$
- $G = \mathfrak{S}_3$, $X = \mathfrak{S}_3$,
$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, x) &\mapsto g \bullet x := g^{-1} \circ x \circ g \end{aligned}$$
- $G = \mathfrak{S}_3$, $X = \{$ the set of subgroups of $\mathfrak{S}_3\}$,
$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, H) &\mapsto g \bullet H := g^{-1} H g \end{aligned}$$
- $G = \mathfrak{S}_4$, $X = \mathfrak{S}_4$,
$$\begin{aligned} \bullet \colon G \times X &\to X \\ (g, x) &\mapsto g \bullet x := g \circ x \end{aligned}$$

(2) Let X be a non-empty set admitting an action of a group G.

- Show that the set $\mathrm{Fix}(G) := \{g \in G \mid g \bullet x = x \text{ for all} x \in X\}$ is a subgroup of G
- Show that $\mathrm{Fix}(G) = \cap_{x \in X} \mathrm{Stab}(x)$.

(3) Consider the map :
$$\begin{aligned} \bullet \colon \mathrm{GL}(2, \mathbf{R}) \times \mathbf{R}^2 &\to \mathbf{R}^2 \\ (\gamma, (x, y)) &\mapsto \gamma \bullet (x, y) := \gamma \cdot \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

- Show that the above map defines an action of $\mathrm{GL}(2, \mathbf{R})$ on $\mathbf{R}^2$.
- What is the orbit of $(1, 0)$?
- What is the stabilizer of $(1, 0)$?

(4) Let G be a finite group. If G acts on X transitively[1] then we have
$$\sum_{g \in G} \mathrm{Fix}(g) = |G|;$$
where $\mathrm{Fix}(g) := \{x \in X \mid g \bullet x = x\}$.

(5) We say that the action of G on X is *doubly transitive* if given any $x_1, x_2 \in X$ and $y_1, y_2 \in X$, there is an element $g \in G$ so that $y_1 = g \bullet x_1$ and $y_2 = g \bullet x_2$.

[1] Recall that a group action is transitive if $|X/G| = 1$, i.e. the action has only one orbit, or equivalently, for any $x, y \in X$ there is a $g \in G$ so that $y = g \bullet x$.

- Show that the following map defines an action (called the diagonal or componentwise action) of G on $Y = X \times X \setminus \{(x, x) \,|\, x \in X\}$:

$$\bullet \colon G \times Y \to Y$$
$$(g, (x_1, x_2)) \mapsto (g \bullet x_1, g \bullet x_2)$$

- Show that an action is doubly transitive if and only if G the above action of G on Y is transitive.

(6) Let G be a group and H a subgroup of G. Consider the action of G on $X = G/H$ by multiplication from left.
- Show that this action is transitive.
- Show that the kernel of the action, that is $\ker(\pi)$ where $\pi\colon G \to \mathfrak{S}(G/H)$ is the associated permutation representation, is the largest normal subgroup of G contained in H.

(7) Let G be a finite group of cardinality n and p be the *smallest* prime number dividing n. Show that if H is a subgroup of G so that $|G/H| = p$, then H is a normal subgroup. Hint: Take such subgroup H and consider the action of G on G/H. What can you say about the kernel of this action?

(8) Consider the quaternion group $Q_8 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$.
- Show that Q_8 is isomorphic to a subgroup of $\mathfrak{S}_8$. Hint: Find a subgroup H of Q_8 so that the action of Q_8 on Q_8/H has trivial kernel.
- Show that Q_8 cannot be realized as a subgroup of any $\mathfrak{S}_n$ if $n \leq 7$. Hint: Assume to the contrary that Q_8 is isomorphic to a subgroup of $\mathfrak{S}_n$, $n \leq 7$. Show that Q_8 then acts on the set $X = \{1, 2, \ldots, n\}$ and deduce that the stabilizer of any $x \in X$ must contain $\langle -1 \rangle$.

(9) What does the class equation say about abelian groups?

(10) Determine each factor in the class equation for following groups :
- $G = \mathfrak{S}_3$
- $G = \mathfrak{S}_4$
- $G = D_8$
- $G = Q_8$

(11) In this exercise, we will determine conjugacy classes in $\mathfrak{S}_n$.
- Let $\sigma = (a_1\ a_2\ \ldots\ a_k)$ be a cycle in $\mathfrak{S}_n$ and $\tau \in \mathfrak{S}_n$ be an arbitrary element. Show that

$$\tau\sigma\tau^{-1} = (\tau(a_1)\,\tau(a_2)\ \ldots\ \tau(a_k))$$

- Generalize the previous exercise to an arbitrary element of $\mathfrak{S}_n$.
- Deduce that two elements in $\mathfrak{S}_n$ are conjugate if and only if they have the same cycle type[2]
- Show that the number of conjugacy classes in $\mathfrak{S}_n$ is equal to the number of partitions of n[3].

(12) Let G be a group of order $2k + 1$. Show that if $x \in G \setminus \{e\}$ then x and x^{-1} cannot be conjugate.

(13) Two subgroups H_1 and H_2 are said to be conjugate in G if there is some $g \in G$ so that $gH_1g^{-1} = H_2$. In this case, we say that H_1 and H_2 are conjugate.
- Decide whether the above relation defines an equivalence relation on the set of subgroups of G.
- Show that the number of conjugates of a subgroup H of G is the index of its centralizer, that is $[G : C_G(H)]$.

(14) Let p be a prime number and G a group of order p^n for some positive integer n. Show that G must have non-trivial center.

(15) Let G be a group of order p^2. Show that G is abelian.

(16)

[2] Any element σ of $\mathfrak{S}_n$ can be written as a product of disjoint cycles of length n_i, for $i = 1, 2, \ldots k$. The non-decreasing sequence of integers $n_1 \leq n_2 \leq \ldots \leq n_k$ is called the cycle type of σ.

[3] A partition of n is a non-decreasing sequence of integers whose sum is n