MATH 504
EXERCISES 6

A. ZEYTİN

Unless otherwise stated G and G' are groups and X is a non-empty set admitting an action of G.

(1) Determine n_2 in $\mathfrak{S}_5$

(2) Show that if G is a group of order 80 then either $n_5 = 1$ or $n_2 = 1$.

(3) Let G be a non-cyclic group of order 57. Determine the number of elements of order 3 in G.

(4) Show that a group of order 45 is necessarily abelian.

(5) Show that groups of following orders are not simple :
- 132
- 200
- 462
- 1365
- 2907
- 6545
- $2^4 5^6$

(6) Let G be a group of order 105. Show that if $n_3 = 1$ then G is abelian.

(7) Let G be a group of order 231. Show that
- G has only one Sylow 7 and only one Sylow 11 subgroup.
- G has a cyclic subgroup of order 33.
- Let P be the Sylow 11 subgroup of G. Show that $P \leq Z(G)$. Hint: As we did in lecture, establish a homomorphism between the automorphism group of a Sylow 11 subgroup P, i.e. $\mathrm{Aut}(P)$, and G.

(8) Let G be a group with $|G| = 315$ and $n_3 = 1$.
- Show that if $\{P\} = \mathrm{Syl}_3(G)$, then $P \leq Z(G)$. Hint: As we did in lecture, establish a homomorphism between the automorphism group of the Sylow 3 subgroup P, i.e. $\mathrm{Aut}(P)$, and G.
- Deduce that G is abelian.

(9) Determine the number of elements of order 7 in a group G of order 168, assuming G is simple. Give an example of a non-simple group of order 168.

(10) Let p be a prime number and let $n \in \{p, p+1, \ldots, p^2 - 1\}$. Find an element H of $\mathrm{Syl}_p(\mathfrak{S}_n)$. Is H abelian?

(11) Let G be a finite group and assume that for each prime number p dividing $|G|$ $n_p = 1$. Set $H_p \in \mathrm{Syl}_p(G)$
- Show that for distinct prime numbers p and q if $x \in H_p$ and $y \in H_q$ then $xy = yx$.
- Deduce that if $p_1, \ldots, p_k$ is the list of prime numbers dividing $|G|$, then $G \cong H_{p_1} \times H_{p_2} \times \cdots H_{p_k}$.
- Deduce that if G is a group of order 99 then $G \cong H_9 \times H_{11}$.
- Find all subgroups of order 99.