MATH 504
EXERCISES 7

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Unless otherwise stated G and G' are groups and X is a non-empty set admitting an action of G.

(1) Show that for any $i \in \mathbf{Z}_+$, $C_i(G \times G') \cong C_i(G) \times C_i(G')$. Deduce that if G and G' are nilpotent groups then so is $G \times G'$.

(2) Show that the group given by the presentation

$$\langle a, b, c \mid a^3 = b^3 = c^4 = acac^{-1} = aba^{-1}bc^{-1}b^{-1} = e \rangle$$

defines the trivial group.

(3) Show that the group given by the presentation

$$\langle a, b \mid ab^2a^{-1}b^3 = ba^2b^{-1}a^{-3} = e \rangle$$

defines the trivial group.

(4) Let G be a group. For two elements $a, b \in G$ the element $[a, b] = aba^{-1}b^{-1}$ is called the commutator of a and b.

- Show, by an example, that the set of commutators, that is $\{[a, b] \mid a, b \in G\}$ does not form necessarily a subgroup of G.
- By G' we denote the subgroup *generated* by the set of commutators : $G' = \langle [a, b] \mid a, b \in G \rangle$. G' is called the *first derived subgroup* of G. Determine G' for
  - i. G an abelian group,
  - ii. $G = \mathfrak{S}_3$
  - ii. $G = \mathfrak{S}_4$
  - iii. Q_8
  - iv. $G = D_{2\cdot 4}$
- Define inductively the i^{th} *derived subgroup* of G, namely $G^{(i)}$, as $(G^{(i-1)})'$. Notice that $G^{(i+1)} \leq G^{(i)}$. Compute $G^{(i)}$ for each $i \in \mathbf{Z}_+$ for the following subgroups :
  - i. G an abelian group,
  - ii. $G = \mathfrak{S}_3$
  - ii. $G = \mathfrak{S}_4$
  - iii. Q_8
  - iv. $G = D_{2\cdot 4}$
- A group is called solvable (or soluble) if $G^{(i)} = \{e\}$ for some $i \in \mathbf{Z}_+$. Determine which of the groups in the above list is solvable.
- Show that any subgroup of a solvable group is solvable.
- Show that if $\varphi \colon G \to H$ is a group homomorphism and if G is a solvable group, then $\varphi(G)$ is a solvable group.
- Suppose that G is a group, N is a normal subgroup. Show that if both N and G/N are solvable groups, then G is a solvable group.